Dec 1882

MATHEMATICAL TRACTS.

No. III.

INVARIANTS.

Morris Plains, N.J.
Nov. 22d 1882

Prof. Geo. Davidson,

Dear Sir: I [illegible]
the delayed & devious
course of your letter
recd to-day - thanks.
No. III goes to your
address to-day.

Very Respectfully
W. J. [illegible]

Dec 3d

[illegible]

TRACTS

Dec 1882

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 3.

INVARIANTS.

BY

REV. W. J. WRIGHT, PH.D.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

"Τοῦ γὰρ ἀεὶ ὄντος ἡ γεωμετρικὴ γνῶσίς ἐστιν."
PLATO, Rep. VII., 527, *b*.

LONDON:

C. F. HODGSON & SON, GOUGH SQUARE,

FLEET STREET.

1879.

My acknowledgments are due to R. Tucker, Esq., M.A., Honorary Secretary of the London Mathematical Society, for valuable assistance rendered in passing these sheets through the press.—W. J. W.

CONTENTS.

CHAPTER I.

	PAGE
Symmetric Functions of the Differences of Roots	7
Eliminant by Symmetric Functions	12
Discriminants	16

CHAPTER II.

Invariant of the Binary Cubic	21
Covariants	24
Emanants	25
Contravariants	28

CHAPTER III.

Canonical Forms	39
Canonizants	41
Combinants	44
Tact-Invariants	46
Absolute Invariants...	48
Series of Covariants...	50

CHAPTER IV.

Computation of Invariants...	56
Self-Conjugate Triangle	57
Locus of the Intersection of Normals...	60
Equation of the Four Common Tangents	67
Theory of Foci	74

PREFACE TO TRACT NO. III.

THIS Tract takes up the general Theory of Invariants.

It is published in pursuance of a purpose, announced in the first number of this series, to give an account of the principal new methods, processes, and extensions which, since 1841, have been introduced into the study of Mathematics. The chief requisite to this undertaking, which undoubtedly is one of considerable magnitude, is evidently a sufficiently comprehensive reading upon these various subjects.

The English, German, French, and Italian Mathematicians have contributed to their journals and learned societies innumerable memoirs and treatises, whose value and bearing upon the matter in hand the reader cannot determine without some degree of careful examination. It also frequently happens that the time consumed in tracing a fugitive paper is in inverse ratio to its importance.

The reader who wishes to read fully upon this Theory may adopt one of two methods. He may begin at the beginning, reading in order of time the papers of its chief authors and expounders, commencing with the essay of the late Dr. Boole in the Cambridge Mathematical Journal for 1841, and follow this with the numerous papers of living authors—Sylvester, Cayley, Hermite, and Salmon—papers extending through the subsequent volumes of the Cambridge and the Cambridge and Dublin Mathematical Journals, and the Philosophical Magazine, together with the various contributions of Clebsch and Aronhold, and others, in *Crelle*, from Vol. 39 to Vol. 69. Or he may take a reverse course, beginning with the Lessons of Salmon, and those of Serret, on Modern Higher Algebra, which, as compends of this and connecting Theories, are in the main works of great excellence, though oftentimes not as clear and satisfactory as could be desired, or as full and explicit as may be found elsewhere; and then he may extend his reading to the Journals above mentioned, together with the proceedings of the contemporaneous societies — as the Philosophical Transactions, *Comptes Rendus*, &c. But, whatever course he may take, he will doubtless never be able clearly to determine to what authorship he is to ascribe some parts and illustrations of the Theory.

The best reading-room for this work, so far as I can judge, after an experience of nearly two years in European libraries, is that of the British Museum.

In view of the extensive literature upon this subject, it may be asked, what can be accomplished by a work of the size of this Tract? Its actual value, evidently, remains to be seen; but I believe that within these pages the reader will find such an account of the Theory as will enable him to gain a knowledge of its principal propositions, and also to judge, from the explained applications, of its real value in Geometry. The computations of Invariants, Chapter IV., will afford such a guide in the various applications that he will probably be at little loss in extending them at his pleasure. I have been desirous of making these calculations so fully, that no one with a fair geometrical knowledge need fail of understanding how each result was obtained. Nowhere else can such work be found in so elementary a form, and for this reason I hope it may prove acceptable to those persons whose time and opportunities of study are somewhat limited, and to those also who are unwilling to obtain and to read the larger works.

In reviewing the notes which I had taken of the principal contributions to this Theory, I found that I had frequently omitted the proper credits, either through sheer neglect, or want of sufficient knowledge; and hence, without attempting to supply these omissions, as could not well be done in the absence of the books and journals, it was concluded to omit them nearly altogether.

The number of persons who have obtained the preceding Tracts of this series, and who have expressed themselves in terms highly favorable to their publication, is deemed sufficient evidence that they are meeting a public want. One thing which was expected has certainly followed,—a goodly number of my countrymen have been awakened to look, for the first time, upon a vast untraversed domain of mathematical knowledge. To these persons, at least, there can be no doubt as to the direction of the goal. It is now clearly and definitely fixed that mathematical researches will, for a long time to come, be mainly conducted through the media of methods and processes, to whose exposition these Tracts are devoted. The time is not far distant, if it has not already arrived, when a knowledge of these subjects will be considered as necessary to the equipment of a mathematician as the Calculus. It is not meant by this, that it is the duty of every mathematician to make a specialty of algebraic forms, either with or without their geometrical interpretation. But it is meant that the modern treatment of the Higher Geometry should be studied as a part of the general preparation necessary to a student of Physical Science.

W. J. W.

CAPE MAY POINT, N.J.;
April, 1879.

INVARIANTS.

CHAPTER I.

PROLEGOMENA.

1. The Theory of Invariants, as will appear, is based upon a knowledge of the General Theory of Equations and several of its later important extensions. Some of these extensions must be stated, because, although perhaps familiar to the reader as commonly or formerly expressed, they may not be easily recognised in their modern dress or terminology; others, because they have no existence outside of their present form.

2. *Symmetric Functions.*—If the general equation be

$$x^n + a_1x^{n-1} + a_2x^{n-2} + a_3x^{n-3} + \&c. = 0,$$

the Newtonian formulæ give us

$$S_1 = -a_1, \quad S_2 = a_1^2 - 2a_2,$$
$$S_3 = -a_1^3 + 3a_1a_2 - 3a_3, \ \&c.,$$

or, as they are written by Hirsch, Cayley, and others,

$$\begin{aligned}
\Sigma\alpha &= -a_1, \\
\Sigma\alpha^2 &= a_1^2 - 2a_2; \quad \Sigma\alpha\beta = a_2, \\
\Sigma\alpha^3 &= -a_1^3 + 3a_1a_2 - 3a_3, \\
\Sigma\alpha^2\beta &= -a_1a_2 + 3a_3, \\
\Sigma\alpha\beta\gamma &= -a_3, \ \&c.;
\end{aligned}$$

in which we have expressed the sum of the roots and the sum of their products by twos, by threes, &c.

3. If we consider any one of these products, as a_1a_2, we say that its weight is $1+2$, or, in general, that the weight of any term is the sum of the suffixes. Looking at these functions,

however far we may extend them, we see that they are symmetrical as to weight. The *order* is estimated by the number of factors in each term. Hence $a_1a_2a_3$ is of the third order, and its weight is $1+2+3$. This being stated, it is easy to see, by inspection of the several functions written above, that the weight of $\Sigma\alpha^t\beta^u$ is $t+u$, and the order the greater of t, u. The order of $\Sigma\alpha\beta\gamma$ (being the sum of the products in threes) can evidently be, so far as the coefficients of the given equation are concerned, only unity. If, therefore, we regard α as the leading root, appearing in every function, we might predicate the degree of the function upon the degree of α. In this case any symmetric function of the p^{th} order must contain more or less terms involving α^p. There will then be p factors each including α. Hence $\Sigma\alpha^3$, $\Sigma\alpha^3\beta\gamma$ are each of the third order in the coefficients of the given equation; that is, the highest order in any term is three. In general, then, the order of any symmetric function is determined by the highest degree in any one root, while the weight is estimated by the total degree of the roots as factors. The literal part, then, of any symmetric function can thus be at once written out. For the sake of clearness, it is necessary to notice that the functions of roots in this manner may be expressed in terms of the coefficients of the given equation, as will be seen by solving the linear equations just written for S_1, S_2, &c.; and, consequently, any function of the differences of roots can be expressed in the same terms.

3. *Symmetric functions of the differences of roots.*—These we shall see are *invariants*. For the present let us consider what relation such functions ought to satisfy. We begin by observing the effect upon the coefficients of the given equation of increasing or diminishing all the roots by the same quantity. There will plainly be no change in the resulting functions of the differences of roots. Let then $x+l$ be substituted for x, and we have

$$x^n+(a_1+nl)\,x^{n-1}$$
$$+[a_2+(n-1)\,la_1+\tfrac{1}{2}n\,(n-1)\,l^2]\,x^{n-2}+\&c.=0.$$

Next, observe the form of any function f of the coefficients a_1, a_2, a_3, &c., when a_1, a_2, &c. are changed into a_1+da_1, a_2+da_2, &c.

This form will be

$$f+\left[\frac{df}{da_1}\,da_1+\frac{df}{da_2}\,da_2+\&c.\right]+\frac{1}{1.2}\left[\frac{d^2f}{da_1^2}\,(da_1)^2+\&c.\right]\ldots(1).$$

But, by the substitution of $x+l$ for x, a_1 becomes a_1+nl, and a_2 becomes $a_2+(n-1)\,la_1+\frac{1}{2}n\,(n-1)\,l^2$.

Clearly, then, if this substitution were made in any function of the coefficients a_1, a_2, &c., and the result arranged with reference to l, we must have, by (1),

$$f+l\left[n\,\frac{df}{da_1}+(n-1)\,a_1\,\frac{df}{da_2}+(n-2)\,a_2\,\frac{df}{da_3}\right]+\&c.=0.$$

This is true whatever l may be.

Let $l=0$, and we have, as the condition which any function of the differences will satisfy,

$$n\,\frac{df}{da_1}+(n-1)\,a_1\,\frac{df}{da_2}+(n-2)\,a_2\,\frac{df}{da_3}+\&c.=0.$$

This relation is both necessary and sufficient in order that the given function of the coefficients should remain unchanged by the substitution of $x+l$ for x in the given equation.

We can now write, not only the literal part, but the coefficients of any symmetric function. For instance, if we are to form $\Sigma\,(\beta-\gamma)^2$, we see that its order is 2 and its weight 2. There can be no more than two factors in any term, while the weight for each term must be 2. It must be of the form $Aa_2+Ba_1^2$. By the above differential equation,

$$[A\,(n-1)+2nB]\,a_1=0.$$

This gives $B=-\dfrac{n-1}{2n}$, when $A=1$; or the function can differ by only a factor from $(n-1)\,a_1^2-2na_2$.

We may see that this factor is unity by supposing $\gamma=1$ and the other roots 0; then $a_2=0$; and $a_1=1$, since $\alpha+\beta+\&c.=a_1$, and $\alpha(\beta+\gamma+\&c.)+\beta\gamma+\&c.=a_2$.

4. The homogeneous equation

$$(a_0,\ a_1\ \ldots\ldots\ a_n)(x,\ y)^n$$

or $$a_0x^n+na_1x^{n-1}y+\tfrac{1}{2}n(n-1)a_2x^{n-2}y^2+\ldots\ldots+a_ny^n=0$$

reduces to the general equation of Art. 2 by dividing by a_0y^n. And it is plain that the differential equation of the last Article will undergo a corresponding change. Hence, for the substitution of $x+l$ for x, we must write

$$a_0\frac{df}{da_1}+2a_1\frac{df}{da_2}+3a_2\frac{df}{da_3}+\&c.=0,$$

while for the substitution $y+l$ for y it must be written in a reverse order, that is,

$$na_1\frac{df}{da_0}+(n-1)\,a_2\frac{df}{da_1}+(n-2)\,a_3\frac{df}{da_2}+\&c.=0.$$

5. *The symmetric function of the homogeneous equation in* $\dfrac{x}{y}$.—Suppose a one of the roots, then $\dfrac{x}{y}=a$. That is, any system of values, as $\dfrac{x_1}{y_1}=a$, in other words, any ratio which is $=a$, will satisfy the homogeneous equation. Or, we may state it thus: any symmetric function expressed in terms of its roots, as x_1, x_2, x_3, &c., may be reduced to the corresponding function of a homogeneous equation of the same degree, by dividing each x_1, x_2, &c. by y_1, y_2, &c., and then multiplying this result by any power of y_1y_2, &c. that will clear it of fractions.

Hence we may write any function of the differences as the sum of products of determinants

$$\Sigma\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix}\ \&c.\times(y_1y_2\&c.)^n,$$

where $n =$ the variable power necessary to clear of fractions. Thus, to form for $(a, b, c, d)(x, y)^3$ the sum of the products of the squares of the differences of the roots, we have the ratios, or roots,

$$\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3},$$

that is, $$\Sigma \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}^2 \times \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}^2 \times \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}^2.$$

In this case the order is 4 and weight 6.

The form is therefore

$$Aa_3^2 a_0^2 + Ba_3 a_2 a_1 a_0 + Ca_3 a_1^3 + Da_2^3 a_0 + Ea_2^2 a_1^2 \;\ldots\ldots\; (1).$$

Operating with $a_0 \dfrac{d}{da_1} + 2a_1 \dfrac{d}{da_2} + 3a_2 \dfrac{d}{da_3}$, we get

$$(B+6A)\, a_3 a_2 a_0^2 + (3C+2B)\, a_3 a_1^2 a_0$$
$$+ (2E+6D+3B)\, a_2^2 a_1 a_0 + (4E+3C)\, a_2 a_1^3 = 0,$$

which gives us, taking $A = 1$, and equating each term to 0,

$$B = -6, \quad C = 4, \text{ \&c.};$$

or, since $a_0 = a$ and $a_3 = d$, we have

$$a^2d^2 - 6abcd + 4b^3d + 4ac^3 - 3b^2c^2.$$

The result would be the same had we required the product of the squares of the differences

$$(\alpha-\beta)^2 (\beta-\gamma)^2 (\gamma-\alpha)^2.$$

The order being the same, 4, and the weight also 6, the form would be

$$Aa_3^2 + Ba_3 a_2 a_1 + Ca_3 a_1^3 + Da_2^3 + Ea_2^2 a_1^2.$$

To render this homogeneous, as if derived from a homogeneous equation in x, y, each factor must be divided by a_0, and the whole multiplied by the highest power of a_0 in any denominator. It would then be identical with (1).

6. The eliminant* or resultant of a system of equations is that function of the coefficients whose vanishing expresses that the equations are simultaneous. If we have as many independent equations as we have variables, we can ordinarily, by direct elimination, arrive at such a function freed from any of the assumed variables. This function is generally indicated by Δ.

7. *Eliminant by symmetric functions.*—The product of the several roots of an equation is a symmetric function, as $\Sigma\alpha\beta\gamma$ or $\Sigma\alpha^2\beta$. If we have α, β, γ as the roots of the equation $f(x)=0$, and α, β_1, γ_1 as the roots of $f_1(x)=0$, then, since they have a common root α, the eliminant condition is involved.

If the first set of roots be substituted in the second equation, $f_1(x)$, the result for the value α will vanish; therefore the continued product

$$f_1(\alpha)\times f_1(\beta)\times f_1(\gamma)$$

will vanish; and consequently will conform to the definition of an eliminant, since it is plain, being a symmetric function of the roots of $f(x)$, it can be expressed in terms of the given coefficients of $f(x)=0$ and $f_1(x)=0$, however they may be written.

From this it is seen that the eliminant is a function of the differences of the roots of the two or more equations.

If the equations are homogeneous, $f(x, y)=0$, $f_1(x, y)=0$, they may be treated as non-homogeneous by dividing each equation by the coefficient of the highest power of x and the highest power of y. To illustrate this form of operation, let us find the eliminant of

$$ax^2+2bxy+cy^2=0 \quad\ldots\ldots\ldots\ldots(1),$$

$$a_1x^2+2b_1xy+c_1y^2=0 \quad\ldots\ldots\ldots\ldots(2),$$

---

* Thus $\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}=0$, $\begin{vmatrix} a & c \\ a_1 & c_1 \end{vmatrix}^2-\begin{vmatrix} b & c \\ b_1 & c_1 \end{vmatrix}\times\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix}=0$

are determinant expressions for the eliminants of

$$\begin{matrix} ax+b=0 \\ a_1x+b_1=0 \end{matrix} \quad\text{and}\quad \begin{matrix} ax^2+bx+c=0 \\ a_1x+b_1x+c_1=0 \end{matrix} \quad\text{respectively.}$$

or, written in the non-homogeneous form,

$$z^2+nz+n_1=0, \quad z^2+mz+m_1=0.$$

The symmetric function is then

$$(a^2+ma+m_1)(\beta^2+m\beta+m_1)=0,$$

or

$$a^2\beta^2+ma\beta(a+\beta)+m_1(a^2+\beta^2)+m^2a\beta+mm_1(a+\beta)+m_1^2=0,$$

But $a^2+\beta^2=n^2-2n_1$ (Art. 2), $a\beta=n_1$, $a+\beta=-n$.

Hence $$(n_1-m_1)^2+(m-n)(n_1m-nm_1)=0.$$

Giving m, n, m_1, n_1 their values, we have

$$\left(\frac{c}{a}-\frac{c_1}{a_1}\right)^2+\left(\frac{2b_1}{a_1}-\frac{2b}{a}\right)\left(\frac{2b_1c}{aa_1}-\frac{2bc_1}{aa_1}\right)=0,$$

or $$(ca_1-c_1a)^2+4(b_1a-ba_1)(b_1c-bc_1)=0,$$

the eliminant. This method is useful in this place simply as an exercise in symmetric functions. In practice, it would be far easier to eliminate directly.

8. *The order.*—By inspecting this example and others, we are enabled to determine inductively the order of the eliminant in the coefficients. The symmetric function consists of as many factors as there are units in the degree of the first equation, but each of these factors involves the coefficients of the second in the first degree. On the other hand, the entire product consists of the several symmetric functions of the roots of the first equation, and the highest degree of these is the same as that of the second equation; hence it is evident that the orders of the coefficients in the eliminant are the same as those of the given equations, but taken in an inverse order; that is, the coefficients of the first equation have the order of the second, and the contrary.

If, for instance, there were three homogeneous equations in three variables of the 2nd, 3rd, and 4th orders, then the eliminant would be a homogeneous function of the 12th order in the coefficients of the first equation, of the 8th in those of the second, and of the 6th in those of the third.

9. *The weight.*—It is not so easy to determine the weight. But we may begin by considering that the eliminant is a symmetric function of the differences between the roots of the first and second equations expressed in terms of their cofficients, and then the number of these differences is equal to the product of the orders of the equations. If we multiply each root by any factor as k, we do, in effect, multiply each difference by k; and consequently, the eliminant, which is the product of these differences, is multiplied by k to a power equal to the product of the degrees of the equations. Now, each root in the equations

$$a_0x^n + na_1x^{n-1}y + \tfrac{1}{2}n(n-1)\,a_2x^{n-2}y^2 + \&c. = 0 \text{ ...... } (1),$$
$$b_0x^m + mb_1x^{m-1}y + \tfrac{1}{2}m(m-1)\,b_2x^{m-2}y^2 + \&c. = 0 \text{ ... } (2),$$

will, it is evident, be multiplied by k when we multiply a_1, b_1; a_2, b_2; by k, k^2, &c.; and therefore each term of the eliminant would involve k to the mn^{th} degree. In this manner we can readily determine the weight of each term, which we shall find to be constant, that is, mn for each term.

10. It is easy to see, from the definition of an eliminant and from the results of (Art. 3), that the eliminant must satisfy the differential equations there given; or, if referred to equations (1) and (2) of the last Article, must be of the form

$$a_0\frac{d\Delta}{da_1} + 2a_1\frac{d\Delta}{da_2} + 3a_2\frac{d\Delta}{da_3} + \&c. + b_0\frac{d\Delta}{db_1} + \&c. = 0,$$

where Δ represents the eliminant of equations (1) and (2).

11. *The eliminant of three equations in three variables.*—The eliminant vanishing, the equations are simultaneous. This can be brought under the system of two equations. For, solving between any two equations, and substituting these values in the third, the product of these substitutions must vanish, since, by hypothesis, there is a community of values between the

different sets, the number of which must equal the weight of the eliminant of those two equations, that is, the product of their degrees. These substituted successively in the remaining equation, and multiplied together, will furnish the requisite symmetric functions by which the coefficients of the solved equations may be expressed, which gives the eliminant whose weight is equal to the product of the degrees of the three equations. For four equations we proceed in the same manner, solving for three and substituting these values in the fourth.

12. In reviewing this method of elimination, it will be seen to be of the widest generality, and all its results susceptible of very satisfactory proof. It is not introduced for any use in actual elimination, but that the reader may here avail himself of important assistance in the study of the Theory of Invariants.

13. The reader interested in determinants will naturally seek some form for elimination by this method. That of Euler leading in this direction is of high theoretical value. Two equations, homogeneous or otherwise, are supposed to be satisfied by a common root of the first degree; then the first, multiplied by all the remaining factors of the second, is evidently equal to the second multiplied by all the remaining factors of the first; as, if we have

$$x^2-(a+b)\,x+ab = 0,$$
$$x^2-(a+c)\,x+ac = 0,$$

then $(x-c)\,\{x^2-(a+b)\,x+ab\} = (x-b)\,\{x^2-(a+c)\,x+ac\}$;

or, in general, if we multiply the homogeneous equation $f(x, y) = 0$ by any arbitrary function of a degree one less than $f_1(x, y) = 0$, and the latter by any arbitrary function with a degree one less than the former equation, and then equate term to term, we shall have a number of equations equal to the sum of the degrees of the two given equations, and the eliminant will of course appear in the form of a determinant.

To eliminate between

$$ax^2+2bxy+cy^2,$$
$$a_1x^3+3b_1x^2y+3c_1xy^2+dy^3:$$

$$(kx^2+klxy+ly^2)(ax^2+2bxy+cy^2)$$
$$=(k_1x+l_1y)(a_1x^3+3b_1x^2y+3c_1xy^2+dy^3);$$

equating like terms,

$$ka-k_1a_1=0,$$
$$2bk+kla-3k_1b_1-l_1a_1=0,$$
$$kc+la+2klb-3k_1c_1-3l_1b_1=0,$$
$$2lb+klc-k_1d-3l_1c_1=0,$$
$$lc-l_1d=0.$$

Eliminating k and l, we have

$$\begin{vmatrix} a & 0 & 0 & -a_1 & 0 \\ 2b & 0 & a & -3b_1 & -a_1 \\ c & a & 2b & -3c_1 & -3b_1 \\ 0 & 2b & c & -d & -3c_1 \\ 0 & c & 0 & 0 & -d \end{vmatrix}=0$$

as the eliminant.

14. The various other methods, such as Bezout's method,* Sylvester's dialytic process, the uses of the Jacobian in elimination, explained in (D. 39),† since they do not illustrate the Theory of Invariants, may be omitted.

We will now pass at once to a subject which is intimately connected with that theory.

15. *Discriminants.*—If an equation, or quantic, as it is called when it is not equated to 0, be differentiated with

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* I must qualify this statement, so far as it relates to Bezout's method. It is well known by those acquainted with Dr. Sylvester's researches, that what he calls a Bezoutiant is the discriminant of a quadratic function in any number of variables, and is expressible as a symmetrical determinant which is written, as in (D. 22), with a double suffix. The eliminant of two equations of the n^{th} degree may be similarly expressed. The use of the Bezoutiant in the theory of equations is exhibited in a Memoir by Sylvester, *Phil. Trans.*, 1853, p. 513.

† Tract No. 1, Determinants.

respect to its variables, the eliminant of these several differentials is the discriminant. As the quantic is understood to be homogeneous, it is evident that the discriminant must be homogeneous also. The order of the discriminant is clearly the product of the degrees of the differentials of which it is the eliminant.

Observing the same order of the suffixes a_0, a_1, &c., the weight of the discriminant will depend upon the number of differentials and the order of the quantic. Thus, for a binary quadratic the weight must be 2; for a ternary cubic, that is, a quantic containing three variables, $3(3-1)^2$. This arises from a slight modification of the reasoning in Art. 9. The weight would be evidently $(n-1)$, taken as many times as a factor as there are variables, were it not for the consideration that all but one of these differentials begin with a coefficient whose relation to the leading variable is the same as in the original quantic; in other words, with a suffix one greater than the first differential which begins with a_0; hence the number of suffixes must be increased in this proportion. If $p =$ the number of differentials, $(n-1)^p$ must be increased by $(n-1)^{p-1}$, that is,

$$(n-1)^p+(n-1)^{p-1} = n(n-1)^{p-1},$$

which is the sum of the suffixes for each term of the discriminant.

16. If we divide the homogeneous equation by y^n, the result is reducible to a product of factors, as

$$\begin{vmatrix} x & y \\ x_1 & y_1 \end{vmatrix} \times \begin{vmatrix} x & y \\ x_2 & y_2 \end{vmatrix} \cdot \times \begin{vmatrix} x & y \\ x_3 & y_3 \end{vmatrix} \times \text{\&c.} = 0.$$

Comparing this product with

$$a_0x^n+na_1x^{n-1}+\text{\&c.} = 0,$$

we see that $$y_1y_2y_3 \text{ \&c.} = a_0,$$

since the product of

$$(xy_1-x_1y)(xy_2-x_2y)(xy_3-x_3y) \text{ \&c.} \equiv Q \text{ .... (1)}$$

gives y_1y_2 &c. for the coefficient of x^n.

17. The discriminant is equal to the continued product of the squares of the differences of the roots of the given quantic taken two and two.

Suppose x_1y_1, x_2y_2, x_3y_3, &c. are the roots of (1) above,

then $\dfrac{dQ}{dx} = y_1(xy_2 - yx_2)(xy_3 - yx_3)$ &c. $+ y_2(xy_1 - x_1y)$ &c.

Observing the effect of substituting x_1y_1 in $\dfrac{dQ}{dx}$, which is $y_1(x_1y_2 - y_1x_2)$ &c.; substituting in the same manner x_2y_2, x_3y_3, &c. in the same equation, and taking the continued product, we must have

$$y_1y_2 \text{ \&c. } (x_1y_2 - y_1x_2)^2 (x_1y_3 - y_1x_3)^2 \text{ \&c.} = 0 \text{.........(1)},$$

which, as we have seen, is the eliminant (Art. 7) of Q and $\dfrac{dQ}{dx}$. The same product, divided by y_1y_2 &c. $= a_0$, will give the discriminant.

This will more fully appear when we consider that

$$nQ = x\frac{dQ}{dx} + y\frac{dQ}{dy};$$

then, when we have substituted successively all the roots of $\dfrac{dQ}{dx} = 0$ in Q, we shall have for the continued product $y_1y_2y_3$ &c. multiplied by a similar result of substituting the same roots in $\dfrac{dQ}{dy}$. But this latter result is evidently the discriminant. Hence, if (1) be divided by a_0, that is, if the eliminant of the quantic and its first differential with reference to x be divided by the product of the y's, we shall obtain the same result as if we had found the eliminant of the first differentials with reference to x and y.

18. Enough preliminary matter has now been introduced to enable the reader to follow with profit all that will follow.

To those who wish to pursue the theory of discriminants further, and desire to study an interesting geometrical application, the theorem of Joachimsthal, taken as the basis of an investigation on the nature of cones circumscribing surfaces having multiple lines, by Dr. Salmon ("Cambridge and Dublin Math. Journal," 1847 and 1849) would probably prove as fruitful in this direction as any that could be mentioned.*

* The theorem above alluded to is included in the following statement. If we have the quantic $(a_0, a_1 \dots a_{n-1}, a_n \between x, y)^n$, and a_1 contain a factor t, and if a_0 contain t^2 as a factor, the discriminant will be divisible by t^2; also, if a_2 contain t as a factor, and if a_1 and a_0 contain t^2 and t^3 respectively, then the discriminant will be divisible by t^6, and so on. The application by Dr. Salmon was that, if $a_0 + a_1x + a_2x^2 +$ &c. be the equation to a surface, and if xy be a double line, a_0 will contain y in the second, and a_1 in the first degree. The discriminant in respect to x is divisible by y^2, and the locus is a tangent cone.

CHAPTER II.

FORMATION OF INVARIANT FUNCTIONS.

19. THE definition of an invariant and covariant of a single quantic has already been given (D. 42). In pursuance of this, we might proceed at once to show how in general such functions can be formed, and then give some explanation of the geometrical importance of the theory; but, for the sake of clearness, we will commence with one of the simplest examples of an invariant function.

20. The determinant of a system of linear equations is an invariant of that system, because, as it will be remembered, when the variables are all transformed by the same linear substitution, the determinant of the transformed equations is equal to the determinant of the given equations multiplied by the modulus of transformation (D. 42). In other words, the determinant (function of the coefficients) of the given equations, which remains unaltered by the transformation, is called an invariant. The equations of (D. 17) will exactly illustrate this.

When the linear equations

$$\left.\begin{aligned} dv + ev_1 + fv_2 &= 0 \\ d_1v + e_1v_1 + f_1v_2 &= 0 \\ d_2v + e_2v_1 + f_2v_2 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

are transformed by the substitutions

$$\left.\begin{aligned} ax + by + cz &= v \\ a_1x + b_1y + c_1z &= v_1 \\ a_2x + b_2y + c_2z &= v_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

then the determinant of the transformed system will be equal to $(de_1f_2) \times (ab_1c_2)$, which may be written

$$f\,(\text{transformed}) = \Delta^n f\,(\text{given}),$$

where Δ = the modulus = the determinant formed from the sinister members of (2).

The f, which is also a determinant expressing the coexistence of equations (1), is in this case called an *invariant*.

21. It is not difficult to see from the above that a somewhat complicated problem is now presented to us. We are to trace the effect of linear transformation upon the same functions of the coefficients of an equation, to determine the number of the functions which remain unaltered by such transformation, and to deduce convenient rules for their formation. It was seen, for instance (D. 41), that when the binary quadratic $(abc \between x, y)^2$ was linearly transformed by the substitution of

$$x = lx + my,$$
$$y = l_1x + m_1y,$$

we wrote $Ax^2 + 2Bxy + Cy^2$ as the transformed quadratic, in which

$$A = al^2 + 2bll_1 + cl_1^2,$$
$$C = am^2 + 2bmm_1 + cm_1^2,$$
$$B = alm + b\,(lm_1 + l_1m) + cl_1m_1,$$

and from which we obtained

$$AC - B^2 = (ac - b^2)(lm_1 - l_1m)^2.$$

The invariant in this case, $ac - b^2$, is no other than the discriminant of the given quadratic $ax^2 + 2bxy + cy^2$.

22. Proceeding now to the binary cubic $(a, b, c, d \between x, y)^3$, we obtain its discriminant; that is, we find its two differentials, and by direct elimination their eliminant, which is the discriminant, viz.,

$$4\,(bd - c^2)(b^2 - ac) + (ad - bc)^2,$$

which, in form, is the same as that obtained in Art. 7. And here, again, we say that the invariant in this case is no other than the discriminant.

23. Since $ac-b^2$ is an invariant of the quadratic

$$ax^2+2bxy+cy^2,$$

we can, by the introduction of a constant, derive not only two invariants after the analogy of $ac-b^2$, but one other whose constituents are derived from the coefficients of the transformed system.

Thus, if we have the two quadratics,

$$sx^2+2txy+uy^2,$$
$$s_1x^2+2t_1xy+u_1y^2,$$

we may multiply the second by k, an arbitrary constant, and obtain by addition

$$(s+ks_1)\,x^2+2\,(t+kt_1)\,xy+(u+ku_1)\,y^2,$$

which, by a transformation identical with that in Art. 21, becomes

$$(S+kS_1)\,X^2+2\,(T+kT_1)\,XY+(U+kU_1)\,Y^2;$$

and the invariant, consequently, by symmetry, is

$$(S+kS_1)(U+kU_1)-(T+kT_1)^2$$
$$=\Delta^2\left[(s+ks_1)(u+ku_1)-(t+kt_1)^2\right].$$

Since this equation is satisfied by any value of k, and therefore identical, the coefficients of the like powers of k must be equal, and therefore

$$SU-T^2=\Delta^2\,(su-t^2),$$
$$S_1U_1-T_1^2=\Delta^2\,(s_1u_1-t_1^2),$$
$$SU_1+S_1U-2TT_1=\Delta^2\,(su_1+s_1u-2tt_1).$$

The last of these is therefore an invariant of a system of two quantics.

And here it would be well to observe that, if we had operated upon the given invariant $su-t^2$ with $s_1\dfrac{d}{ds}+t_1\dfrac{d}{dt}+u_1\dfrac{d}{du}$, the result would have been the same as that actually obtained by the substitution of $s+ks_1$ for s, &c.; and thus, in general, if we have an invariant of any known quantic, we may find the invariant of a system of two or more simultaneous quantics of the same degree, either by substitution or by the use of the operator, as above.

24. As we have stated, the binary quadratic and cubic have no other invariant than their respective determinants; but, as we shall see, binaries of a higher degree may have two or more functions unaltered by transformation. For instance, if we take the binary quartic

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4,$$

and operate upon it with the symbols* $\frac{d}{dy}$ and $-\frac{d}{dx}$, that is, substitute $\frac{d}{dy}$ for x and $-\frac{d}{dx}$ for y; we shall find that the result

$$ae-4bd+3c^2$$

will conform to the definition of an invariant. The same will be true if we expand the determinant formed from the fourth differentials of the quartic, viz.,

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = ace+2bcd-ad^2-eb^2-c^3 \ldots\ldots\ldots (1).$$

That these two invariants may be derived from the binary quartic, may be shown by actually transforming the given

* The theory of these symbols must be reserved for another part of the subject; see Arts. 28, 33. The actual process is to introduce the symbols into the quantic, thus obtaining a differential symbol. Thus, by substituting $-\frac{d}{dx}$ for y, and $\frac{d}{dy}$ for x in the given quantic, we have

$$a\frac{d^4}{dy^4}-4b\frac{d^4}{dy^3dx}+6c\frac{d^4}{dy^2dx^2}-4d\frac{d^4}{dy\,dx^3}+e\frac{d^4}{dx^4}.$$

Operating upon the quartic with this symbol, we get

$$48ae-192bd+144c^2.$$

Hence the relation is $\quad ae-4bd+3c^2.$

If the quantic had been of odd degree, the result would have vanished.

quartic, equating the values of A, B, C, &c., and we should find

$$AE-4BD+3C^2=\Delta^4(ae-4bd+3c^2),\ *$$

and also another, (1), which entering into the discriminant forms still a third. These latter, however, do not at present concern us beyond the assurance that they exist, the theory of their formation being reserved for future consideration.

25. The Theory of Covariants will be found to be immediately connected with the Theory of Invariants. This follows from the fact that the covariant is a function not only of the coefficients, but also of the variables of the given quantic; that is,

$$f(S,\,U,\,\&c.\ X,\,Y,\,\&c.)=\Delta^m f(s,\,u,\,\&c.\ x,\,y,\,\&c.)$$

To illustrate this, let us take the Hessian (D. 40) of the quartic

$$ax^4+4bx^3y+6cx^2y^2+4dxy^3+ey^4 \text{ .... } (1),$$

and we have

$$\begin{vmatrix} ax^2+2bxy+cy^2, & bx^2+2cxy+dy^2 \\ bx^2+2cxy+dy^2, & cx^2+2dxy+ey^2 \end{vmatrix} \text{ .... } (2),$$

which, expanded, gives the invariant form $mn-l^2$, and differs in form only from the invariant of the quadratic (Art. 23) $ac-b^2$ by the variables of the given quantic.

Looking at this example a little further, we see that (1) and (2) contain the same powers of the variables, and equally the same coefficients. Hence the invariant of the covariant in this case can be no other than the invariant of (1), and this conclusion may easily be seen to be general.

26. Covariants may be formed by substituting in the given quantic $x+kx_1$ and $y+ky_1$ for x and y. The coefficients of the several powers of k form covariants, and, taken in order, are called emanants of the quantic. Thus, if we take the binary cubic

$$ax^3+3bx^2y+3cxy^2+dy^3,$$

* To transform this quartic, ax^4 &c., the reader has only to repeat the process of Art. 21 on a larger scale.

and substitute as proposed, we shall find the several emanants to be $\left(x_1\frac{d}{dx}+y_1\frac{d}{dy}\right)^n$, in which form the coefficients of the ascending powers of k appear.

The emanants of any quantic can in general be expressed in the form

$$\left(x_1\frac{du}{dx}+y_1\frac{du}{dy}\right)^{1,2\ldots n} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

as first, second, and n^{th} emanants.

If we take the second power, to get the second emanant, we may write

$$x_1^2 m^2 + 2x_1 y_1 mn + y_1^2 n^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

as the result, where m and n represent the differentials $\frac{du}{dx}, \frac{du}{dy}$, and x_1, y_1 are regarded as cogredient to,* or vary as, the given variables. Now it is easy to see that, if we regard (1) or (2) as a function of x_1, y_1, and the original variables as constants, and proceed to form the invariants, these invariants will in turn represent covariants of the given quantic, if we then conceive x, y as variables.

This will appear at once by reference to Art. 23, where we were enabled, after transforming the quadratic, to write

$$SU-T^2=\Delta^2(su-t^2).$$

Transform now (2), and write its invariant, and we shall have

$$\frac{d^2V}{dX^2}\cdot\frac{d^2V}{dY^2}-\left(\frac{d^2V}{dX\,dY}\right)^2=\Delta^2\left[\frac{d^2v}{dx^2}\cdot\frac{d^2v}{dy^2}-\left(\frac{d^2v}{dx\,dy}\right)^2\right] \ldots(3),$$

where $V=$ the transformed, and $v=$ the original quantic.

* To exhibit this, let $(abc\between x, y)^2$ be the given quantic. Making the substitutions, we have

$$k^2(ax_1^2+2bx_1y_1+cy_1^2) \text{ and } 2k\left\{axx_1+b(xy_1+x_1y)+cyy_1\right\}$$

as first and second emanants; but, by hypothesis, x_1, y_1 are cogredient to xy; hence each of the coefficients above resume the quadratic form, in other words, they become identical. Hence there are not, as we shall see, two covariants to the quadratic $(abc\between x, y)^2$, but one (as there is no function of the differences of the roots), and that one must be the quantic itself.

This invariant is now the covariant of the given quantic, as is evident algebraically if we compare (2) with the form for the second emanant,

$$\left(x_1\frac{dv}{dx}+y_1\frac{dv}{dy}\right)^2,$$

in which x, y are first treated as constants, by which supposition we form the invariant, and then as variables, so that the result conforms to the definition of an invariant. It is plain also that (3) is the expanded determinant formed from the quadratic emanant, and in this sense may be regarded as a discriminant of the quadratic function, and is therefore an invariant with the limitation that its variables are regarded as constants. The first emanants of a system of linear equations yield a determinant. Hence in general we may say that the Jacobian (D. 39) of the first emanants of a system of linear equations—that is, the first differentials of these equations regarded as functions of x_1, y_1, z_1—will form a determinant which is a covariant of the system.

27. *Inverse Substitution.*—In Trilinear Coordinates α, β, γ are used ordinarily to express the coordinates of the point. Let now

$$x\alpha+y\beta+z\gamma=0$$

be the equation of a straight line in which α, β, γ are the tangential coordinates of the line, that is, its perpendicular distances from the three points of reference; and x, y, z the perpendicular distances of any point in the line from the three lines of reference, that is, its trilinear coordinates (T. 2, 25).*

By transforming this equation to new axes by linear substitution, it will be seen that, while the trilinear coordinates are transformed by direct substitution, the tangential coordinates are transformed at the same time by the inverse substitution.

Let

$$\begin{aligned} x &= l_1X+m_1Y+n_1Z, \\ y &= l_2X+\text{\&c.}, \\ z &= l_3X+\text{\&c.} \end{aligned}$$

* Tract No. II., Art. 25.

Then the new equation of the line may be written

$$\mathrm{A}X + \mathrm{B}Y + \Gamma Z = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where
$$\begin{aligned} \mathrm{A} &= l_1\alpha + l_2\beta + l_3\gamma, \\ \mathrm{B} &= m_1\alpha + m_2\beta + m_3\gamma, \\ \Gamma &= n_1\alpha + n_2\beta + n_3\gamma. \end{aligned}$$

The two sets of coordinates in this case are said to be contragredient to each other; and in general it may be stated that the tangential coordinates, whether of a line or a plane, will be transformed by a different—that is, inverse—substitution, from the coordinates representing different points. The latter are said to be cogredient, as x, x_1, y, y_1, &c., because transformed by the same substitution; while the former are said to be contragredient, because tranformed by an inverse substitution.

28. This may be stated in another form; for, since the equation which was a function of x, y, z has been transformed to a function of X, Y, Z, the total differential coefficients with respect to the latter are functions of those with respect to the former.

We have, from (1) of the last Article,

$$\alpha\,(l_1X + m_1Y + n_1Z) + \beta\,(l_2X + m_2Y + n_2Z) + \gamma\,(l_3X + m_3Y + n_3Z) = 0;$$

and therefore

$$\frac{d}{dX} = l_1\frac{d}{dx} + l_2\frac{d}{dy} + l_3\frac{d}{dz},$$

since
$$\frac{dx}{dX} = l_1, \text{ \&c.}$$

Comparing x, y, z with $\dfrac{d}{dx}, \dfrac{d}{dy}, \dfrac{d}{dz}$, we see that the substitution which linearly transforms the one will linearly transform the other, but by a reciprocal relation as expressed by the determinants of the coefficients

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \qquad \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}$$

If now $\frac{d}{dx}$, $\frac{d}{dy}$, $\frac{d}{dz}$ vanish, as will happen only when the discriminant of the quantic or system vanishes, then $\frac{d}{dX}$ will necessarily vanish also.

29. The consideration of inverse substitution leads directly to a function well known in geometry as the *contravariant.* For it will be seen at once that, if a quantic which is a function of two sets of variables x, y, z; α, β, γ, be linearly transformed, the function involving the coefficients and the variables, regarded as transformed by the inverse substitution, must be similar to the covariant, but which is called the contravariant; that is,

$$f(A_0, A_1, \&c.\ \mathrm{A}, \mathrm{B}, \Gamma) = \Delta^m f(a_0, a_1, \&c.\ \alpha, \beta, \gamma).$$

It is evident also, from what has preceded, that the contravariant may be deduced in a manner similar to that exhibited in Art. 23. If we take the binary quadratic,

$$ax^2 + 2bxy + cy^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

and combine it with $\quad k(x\alpha + y\beta)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$

we shall have

$$(a + k\alpha^2)\,x^2 + 2\,(b + k\alpha\beta)\,xy + (c + k\beta^2)\,y^2 \ldots\ldots\ldots (3).$$

If now (1) becomes, by linear transformation,

$$AX^2 + 2BXY + CY^2,$$

and (2) becomes $\quad k(X\mathrm{A} + Y\mathrm{B})^2,$

then (3) becomes

$$(A + k\mathrm{A}^2)\,X^2 + 2\,(B + k\mathrm{AB})\,XY + (C + k\mathrm{B}^2)\,Y^2,$$

and the invariant form gives

$$(A+k\mathrm{A}^2)(C+k\mathrm{B}^2)-(B+k\mathrm{AB})^2$$
$$=\Delta^2\left[(a+k\alpha^2)(c+k\beta^2)-(b+k\alpha\beta)^2\right].$$

And since we may equate the coefficients of the like powers of k, we have

$$\mathrm{AB}^2-2B\mathrm{AB}+C\mathrm{A}^2=\Delta^2(a\beta^2-2b\alpha\beta+c\alpha^2)\ \ldots\ldots\ldots(1),$$

that is, $a\beta^2-2b\alpha\beta+c\alpha^2$, differing only by a power of the modulus from the corresponding function of the transformed coefficients and variables α, β, is a contravariant.

In reviewing now three functions thus considered, it will be seen that they all equally possess the property of invariance.

30. In general, when α, β, γ are regarded as contragredient to x, y, z, the contravariant may be expressed, by application of the preceding Article,

$$A_0X^n+\&c.+k(X\mathrm{A}+Y\mathrm{B}+Z\Gamma)^n$$
$$=a_0x^n+\&c.+k(x\alpha+y\beta+z\gamma)^n,$$

and the invariant would be

$$f(A_0+k\mathrm{A}^n,\ A_1+k\mathrm{A}^{n-1}\mathrm{B},\ \&c.)=\Delta^m f(a_0+k\alpha^n,\ a_1+k\alpha^{n-1}\beta,\ \&c.)$$

We have thus to develop the sum of two functions, which, by Taylor's Theorem, gives us for the coefficients of the constant k,

$$\left(\alpha^n\frac{d}{da_0}+\alpha^{n-1}\beta\frac{d}{da_1}+\alpha^{n-1}\gamma\frac{d}{db_1}+\ldots\ldots\right)^r P.*$$

If $r=1$, this formula gives us what has been called the first evectant.

* $P=$ the invariant of the quantic.

To apply this, we know that $ac-b^2$ is the invariant of the quadratic. We have then

$$\Delta^2\left(\alpha^2\frac{d}{da}+\alpha\beta\frac{d}{db}+\beta^2\frac{d}{dc}\right)(ac-b^2)$$
$$=\left(A^2\frac{d}{dA}+AB\frac{d}{dB}+B^2\frac{d}{dC}\right)(AC-B^2),$$

which is identical with (1) of the last Article.

If the quantic be a ternary quadratic, as

$$a_0x^2+a_1y^2+a_2z^2+2a_3yz+2a_4xz+2a_5xy,$$

we shall have for an invariant

$$\begin{vmatrix} a_0 & a_5 & a_4 \\ a_5 & a_1 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix} = a_0a_1a_2+2a_3a_4a_5-a_0a_3^2-a_1a_4^2-a_2a_5^2=P,$$

which is the discriminant of the quantic; whence

$$\left(\alpha^2\frac{d}{da_0}+\beta^2\frac{d}{da_1}+\gamma^2\frac{d}{da_2}+\beta\gamma\frac{d}{da_3}+\alpha\gamma\frac{d}{da_4}+\alpha\beta\frac{d}{da_5}\right)P$$
$$=(a_1a_2-a_3^2)\,\alpha^2+(a_0a_2-a_4^2)\,\beta^2+(a_0a_1-a_5^2)\,\gamma^2$$
$$+2\,(a_4a_5-a_0a_3)\,\beta\gamma+2\,(a_3a_5-a_1a_4)\,\alpha\gamma+2\,(a_3a_4-a_2a_5)\,\alpha\beta$$

is a contravariant, and in geometry expresses the condition that a given line represented by a trilinear equation shall touch the given conic, or, in other words, is the tangential equation of the conic.*

It is to be observed that the discriminant of the first evectant of the second degree can be written as a determinant:

$$\frac{dE}{d\alpha}=0,$$

$$\frac{dE}{d\beta}=0,$$

$$\frac{dE}{d\gamma}=0,$$

* Salmon's "Conics," p. 249.

and may be regarded as the invariant of a system, or of the given contravariant.

It is also to be observed that there are instances of functions involving both x, y, z and α, β, γ, and which do not change by transformation of the quantic, that is,

$$f(A_0, A_1 \ldots X, Y \ldots A, B \ldots) = \Delta^m f(a_0, a_1 \ldots x, y \ldots \alpha, \beta \ldots),$$

which have received the general name of *mixed concomitants.*

That such functions may easily be formed may be seen by examining the covariant (1) of Art. 26.

If we subject it to the operation of finding the coefficients by Taylor's Theorem, we shall have $\left(\alpha^2 \frac{d}{da} + \alpha\beta \frac{d}{db} + \beta^2 \frac{d}{dc}\right) C$, where $C =$ the covariant in question.

31. We may here perhaps interest the reader by introducing an illustration of the geometrical application of invariants. It is well known that when we transform from one rectangular system to another, that $a+b$ and $ab-h^2$ remain unaltered by the transformation. Suppose it were inquired as to the form these quantities take when the transformation is made from rectangular (or oblique) to oblique axes, where a, b, h are constants in the quadratic

$$ax^2 + 2hxy + by^2 \quad \ldots\ldots\ldots\ldots \quad (1).$$

Let the transformation be from axes inclined at the angle ω to axes of any other inclination, as Ω. Then, by making the proper substitutions, (1) becomes

$$AX^2 + 2HXY + BY^2 \text{ (Art. 21).}$$

By symmetry, $x^2 + 2xy \cos\omega + y^2$ would become

$$X^2 + 2XY\cos\Omega + Y^2,$$

as either expresses the square of any point from the origin. Adopting now a method with which we are familiar, we say

$$\begin{aligned} &ax^2 + 2hxy + by^2 + k(x^2 + 2xy\cos\omega + y^2) \\ &\qquad = AX^2 + 2HXY + BY^2 + k(X^2 + 2XY\cos\Omega + Y^2). \end{aligned}$$

If we determine k so that the first side of the equation may become a perfect square, the second will become a perfect square also, that is, k must be one of the roots of

$$k^2 \sin^2 \omega + (a+b-2h \cos \omega) k + ab - h^2 = 0.$$

This value of k will make the left member a perfect square. A similar quadratic will be found in the right-hand member, which will make it also a perfect square. Both members become perfect squares for the same value of k, and are therefore equal.

Equating coefficients of corresponding terms, we have, what we already knew (Art. 5) in form,

$$\frac{a+b-2h \cos \omega}{\sin^2 \omega} = \frac{A+B-2H \cos \Omega}{\sin^2 \Omega},$$

$$\frac{ab-h^2}{\sin^2 \omega} = \frac{AB-H^2}{\sin^2 \Omega}.$$

(This elegant demonstration is due to the late Dr. Geo. Boole. See *Cambridge Math. Jour.*, N. S., VI. 87.)

32. From Art. 28, we learn that x, y, z; α, β, γ sustain a reciprocal relation to each other. The same is to be observed of x, y, z and $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$. The transformation of the former transforms the latter, but by an inverse method. In this way the contravariant is obtained, which, as has been remarked, possesses the property of invariance. If now in the contravariant we substitute $\frac{d}{dx}$, &c., we shall obtain a function which, containing signs of operation, and being itself unchanged by transformation, may be called an operating symbol—a type form which, if applied to the quantic or to its covariants, must give either an invariant or a covariant according as the variables disappear or remain after differentiation.

$c\alpha^2 - 2b\alpha\beta + a\beta^2$ being a contravariant of $ax^2 + 2bxy + cy^2$, we obtain, by applying to the quadratic the operating symbol $c\frac{d^2}{dx^2} - 2b\frac{d^2}{dx\,dy} + a\frac{d^2}{dy^2}$, the invariant $ac - b^2$.

33. Proceeding upon the principle now before us, we are enabled to generate, as will be seen, the three functions considered, by means of simple substitution. Since $f(\beta, -\alpha)$ becomes, by a linear transformation, $\frac{f(B, -A)}{\Delta}$, that is, a contravariant, we have then, in a binary quadratic, only to substitute β and $-\alpha$ for x and y to obtain the contravariant.

If, therefore, we write α with the negative sign, there is no reason why we should not say that $-\alpha$, β are transformed by the same rules as x, y. The symbols $\frac{d}{dx}$, &c., which we regarded as contragredient to x, y, may be with equal reason called cogredient to $-x$, y; and, conversely, $\frac{d}{dy}$, $-\frac{d}{dx}$ may be taken as cogredient to x, y. Hence, if we substitute these symbols in either the quantic or its covariants, we obtain a new set of functions of the same form. The exception is seen in the binary quartics, where, for instance in the quadratic, the substitution gives $4(ac - b^2)$, an invariant.

34. A fuller investigation of the quadratic, in the general theory, will lead to what is perhaps already sufficiently evident, that the quadratic $(a, b, c \between x, y)^2$ has no covariant but the quantic itself.* We have seen that its discriminant is the invariant $ac - b^2$, and its contravariant $c\alpha^2 - 2b\alpha\beta + a\beta^2$; and since $ac - b^2$ is an invariant, we learn, from Art. 26, that the second emanant is a quadratic in x_1, y_1, and its discriminant is a covariant, for a quantic higher than the second degree. We know (Art. 22) that the invariant of

$$(a, b, c, d \between x, y)^3 \qquad \dots\dots (1)$$

is

$$a^2d^2 + 4ac^3 - 6abcd + 4db^3 - 3b^2c^2 \qquad \dots\dots (2).$$

* An invariant being a function of the differences of roots, there can be no such function formed other than the given quantic. See Note, p. 24.

Hence for every quantic higher than the third we have the covariant

$$\left[\left(\frac{d^3}{dx^3}\frac{d^3}{dy}\right)^2 + 4\frac{d^3}{dx^3}\left(\frac{d^3}{dx\,dy}\right)^3 - 6\frac{d^3}{dx^3}\frac{d^3}{dx^2\,dy}\frac{d^3}{dx\,dy^2}\frac{d^3}{dy^3}\right.$$
$$\left. + 4\frac{d^3}{dy^3}\frac{d^3}{dx^2\,dy} - 3\left(\frac{d^3}{dx^2\,dy}\frac{d^3}{dx\,dy^2}\right)^2\right]u.$$

The covariant of (1) may be found by forming the evectant (Art. 30)

$$\left(a^3\frac{d}{da} + a^2\beta\frac{d}{db} + a\beta^2\frac{d}{dc} + \beta^3\frac{d}{dd}\right)P,$$

where $P = (2)$.

Then, by substituting x, y for a, β, we have

$$x^3(ad^2 - 3bcd + 2c^3) + 3x^2y(-acd + 2b^2d - bc^2)$$
$$+ 3xy^2(-abd + 2ac^2 - b^2c) + y^3(a^2d - 3abc + 2b^3).$$

And thus generally for binaries, when any invariant is known.

35. If we take any quantic, and observe the effect of any linear substitution, it is easy to see that its invariant will remain unchanged if for x we substitute y or lx, and y for x. It will be seen that the order or degree of the invariant is still constant, and also that the weight, which is estimated by taking the sum of the suffixes of the factors of the several terms, is constant for each invariant.

If s, s_1, s_2, &c. represent the suffixes before transformation, $n-s$, $n-s_1$, $n-s_2$ &c. will represent the suffixes of the same coefficients after transformation, and we shall have

$$s + s_1 + s_2 \text{ \&c.} = n - s + n - s_1 + n - s_2 \text{ \&c.},$$

or

$$2w = nt,$$

where $w =$ the weight of the suffixes for each term of the coefficients, and $t =$ the degree or order of the invariant. In other words, the weight is $= \frac{1}{2}nt$.

In this way the invariant of any quantic may be written at once, the required degree being known.

If, for instance, an invariant of a binary quartic of the second degree in the coefficients is required, we have

$$w = \tfrac{1}{2}nt = 4.$$

There will be as many terms of the proposed invariant as the sum of two numbers 0 ... 4 inclusive can be written; hence

$$A_0 a_0 a_4 + A_1 a_2 a_2 + A_2 a_1 a_3 \text{ ....(1)}$$

is the required invariant. The values of A_0, &c. will depend upon other considerations. The first is, that an invariant must be a function of the differences of the roots; for it is to be unchanged when we effect the transformation by substituting $x+l$ for x; it must therefore satisfy a differential equation for the function of the differences of the roots, as

$$a_0 \frac{dP}{da_1} + 2a_1 \frac{dP}{da_2} + 3a_2 \frac{dP}{da_3} + 4a_3 \frac{dP}{da_4} + \text{\&c.} = 0 \text{ ......(2)}.$$

The second consideration is, that the coefficients thus obtained are clearly proportional.

Applying then (2) to (1), we have

$$(A_2 + 4A_0)\, a_0 a_3 + (4A_1 + 3A_2)\, a_1 a_2 = 0.$$

Taking $A_0 = 1$, we find the invariant to be

$$a_0 a_4 - 4a_1 a_3 + 3a_2^2;$$

or, using the coefficients of the quartic,

$$ae - 4bd + 3c^2.$$

Thus the differential equation furnishes the conditions to determine the values of A_0, &c.

If the number of conditions is greater than these coefficients, there is no additional invariant; if the same, one more, or one alone; if less, more than one. If we wished to obtain the discriminant of the quartic, which is also an invariant, by this method, or rather if we wished to obtain an invariant of the sixth order in the coefficients, we should find the number of ways in which 12 can be written as the sum of 6 numbers from 0 ... 4, and we should have as many conditions as $\frac{1}{2}nt-1$ or 11 can be written as the sum of 6 numbers from 0 ... 4.

36. We may arrive at the covariant in the same manner.

If n represent the degree of the quantic, n_1 the degree of the covariant, in x and y, and m the degree of x in any term, we have

$$m+s+s_1+s_2 \text{ \&c.} = n_1-m+n-s+n-s_1+n-s_2 \text{ \&c.}$$

Calling $m+s+s_1$ &c. the weight, the equation gives

$$w = \tfrac{1}{2}(nt+n_1).$$

If we wished to form, for instance, the quartic covariant to the quartic of the second degree in the coefficients, we could, instead of taking the Hessian of the quantic, which would give the required covariant, estimate the terms multiplying each variable, since $t=2$, $n=4$, $m=4$, and, if we are concerned with the coefficient of x^4, $n_1=4$.

The weight would then be 6, and hence

$$4+s+s_1=6,$$
$$s+s_1=2.$$

There are therefore two terms multiplying x^4 each of the second degree, that is, a_0a_2 and a_1a_1, or ac and b^2.

In the same manner we find, for the terms which multiply x^3,

$$3+s+s_1=6.$$

Hence the terms are a_0a_3 and a_1a_2, or ad and bc, &c. &c.

Now it will be perceived we do not know how by this process to connect ac and b^2, ad and bc.

To ascertain this relation, let us suppose that

$$A_0x^{n_1}+n_1A_1x^{n_1-1}y+\frac{n_1(n_1-1)}{2}A_2x^{n_1-2}y^2+\text{\&c.} \;\ldots\ldots\; (1)$$

represents the covariant.

Suppose also

$$a_0 \frac{d}{da_1} + a_1 \frac{d}{da_2} \text{ \&c. and } na_1 \frac{d}{da_0} + (n-1)\, a_2 \frac{d}{da_1} \text{ \&c.}$$

to be represented by α and β. If now, in (1), we suppose the same substitution as was made in the original quantic, and that

$$\frac{da_0}{d\alpha} = 0, \quad \frac{da_1}{d\alpha} = a_0, \quad \frac{da_2}{d\alpha} = 2a_1, \quad \frac{da_3}{d\alpha} = 3a_2,$$

then $$\frac{df}{d\alpha} = a_0 \frac{df}{da_1} + 2a_1 \frac{df}{da_2} + 3a_2 \frac{df}{da_3} + \text{\&c.};$$

and we can write, on the supposition that these changes are identical,

$$\frac{dA_0}{d\alpha} = 0, \text{ \&c. as above,}$$

and, for the same reason,

$$\frac{dA_0}{d\beta} = n_1 A_1, \text{ \&c.}$$

Thus, when A_0 is a function of the differences, we can find all the other terms of the covariant; that is, we can, by successive differentiation, pass from one term to the other, and thus, by the use of these two operators, determine the exact form of the coefficients of the covariant. Thus, in the case of the quadratic covariant to the quartic, we found A_0 to be of the form $Aa_0a_2 + Ba_1a_1$, which, operated upon by $\frac{dA_0}{d\alpha}$, becomes $(A+2B)a_0a_1 = 0$. If $A=1$, then $B=-1$, and $A_0 = a_0a_2 - a_1a_1$. Operate upon this latter with $\frac{dA_0}{d\beta}$, which in this case is

$$4a_1 \frac{d}{da_0} + 3a_2 \frac{d}{da_1} + 2a_3 \frac{d}{da_2} + a_4 \frac{d}{da_3},$$

and we get $$2\,(a_0a_3 - a_2a_1) = A_1.$$

Again, operating with $\frac{dA_1}{d\beta}$ upon $a_0a_3 - a_2a_1$, and we have

$$4a_1a_3 + a_4a_0 - 2a_3a_1 - 3a_2a_2 = A_2 = a_4a_0 + 2a_1a_3 - 3a_2a_2.$$

Operating then upon this latter with $\frac{dA_2}{d\beta}$, we obtain

$$2\,(a_1a_4-a_2a_3)=A_3.$$

And finally we have

$$\frac{dA_3}{d\beta}=a_2a_4-a_3a_3=A_4.$$

The covariant then, written fully, is

$$(ac-b^2)\,x^4+2\,(ad-bc)\,x^3y+(ae+2bd-3c^2)\,x^2y^2$$
$$+\,2\,(be-cd)\,xy^3+(ce-d^2)\,y^4.$$

We see, therefore, that A_0 is the source of the covariant, and we can readily write

$$A_0\,x^{n_1}+\frac{dA_0}{d\beta}\,x^{n_1-1}y+\frac{d^2A_0}{d\beta^2}\,\frac{x^{n_1-2}\,y^2}{1\,.\,2}+\frac{d^3A_0}{d\beta^3}\,\frac{x^{n_1-3}\,y^3}{1\,.\,2\,.\,3}+\&c.$$

as the law of derivation.

That A_0 is appropriately called the source is evident from its repeated use, being, in fact, operated upon by each successive differential symbol, as is seen on p. 36.

CHAPTER III.

THEORY OF LEAST OR CANONICAL FORMS.

37. WHEN a quantic has been reduced to the least form in which it can be written, and yet retain its generality, it is said to be reduced to its *canonical form.* The theory has been presented by Dr. Sylvester (see *Philosophical Magazine*, Nov. 1851). The name canonical seems to have been first applied by Hermite. The number of constants remains in most cases implicitly the same.

Since $lx+my$ may be represented by X, and $l'x+m'y$ by Y, a cubic in two variables may be represented by X^3+Y^3. This is evident, as the entire number of constants is implied in this form.

The quadratic $(a, b, c)(x, y)^2$ can be reduced with four constants* to the form x^2+y^2, or to a similar form Az^2+Bt^2 containing the original number. But the binary quadratic in geometrical investigations is so completely manageable in its

* To reduce $2x^2+14x+29$ to the sum of two squares.

We have $$(lx+my)^2+(l'x+m'y)^2$$

as the transformed quadratic, or

$$(x+t)^2+(x+t')^2 = 2x^2+14x+29,$$

where $$t = \frac{m}{l}, \text{ and } t' = \frac{m'}{l'},$$

whence $$t^2+t'^2 = 29,$$

and $$t+t' = 7 \text{ or } t = 2, \ t' = 5,$$

while the coefficient of x is plainly 1, therefore $(x+2)^2+(x+5)^2$ is the expression.

original form that its reduction to a sum of squares is not a matter of much interest.

But the reduction of the cubic is of more practical importance, since, independent of geometrical considerations, the reduction to the sum of two cubes furnishes a method of solution of numerical equations. The cubic

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3$$

becomes, we will suppose, by transformation

$$AX^3 + 3BX^2Y + 3CXY^2 + DY^3,$$

and, remembering that the Hessian

$$\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2$$

gives a covariant which may be transformed in the same manner and into a function of the same constants as before, that is,

$$\Delta^2 \left[\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2\right] = \frac{d^2U}{dX^2} \cdot \frac{d^2U}{dY^2} - \left(\frac{d^2U}{dX\,dY}\right)^2 \quad \ldots\ldots(1),$$

we see that the transformed becomes $ADXY$ when B and C vanish.

Or, since we are simply seeking the factors into which the Hessian may break up when B and C vanish, we may omit the factor Δ^2 (being composed of the constants of transformation), and examine the left-hand member of (1) for the required factors X, Y.

With these conditions, the Hessian cannot differ by more than a factor from XY.

As an illustration, let us take

$$4x^3 + 30x^2 + 78x + 70 = 0 = u.$$

The Hessian is

$$\begin{vmatrix} 2x+5 & 5x+13 \\ 5x+13 & 13x+35 \end{vmatrix} = x^2 + 5x + 6.$$

Taking the factors of this, $x+2$ and $x+3$, we have

$$A(x+2)^3+D(x+3)^3$$

for the determination of A and D by comparison with the given quantic

$$A+\quad D=4,$$
$$8A+27D=70,$$

or $$A=2,\quad D=2.$$

Hence $$2(x+2)^3+2(x+3)^3=u,$$

that is, $(x+2)^3+(x+3)^3$ differs by only a factor from u, and therefore

$$(x+2)+(x+3)=0$$

gives $x=-\frac{5}{2}$ as a root of the given cubic. The other roots of this cubic being imaginary, it is evident that not every cubic can be reduced to this form, since it must differ from one which contains three real factors, or one containing a square factor.

In the latter case, we could evidently express the canonical form of the given cubic by $(lx+my)^2(l'x+m'y)$ or $(x+t)^2(x+t')$ or x^2y.

To reduce $x^3+7x^2+16x+12$ to the form x^2y.

We have $$x^3+(t'+2t)\,x^2+(2tt'+t^2)\,x+t^2t',$$

whence $$t'+2t=7,$$
$$2tt'+t^2=16,$$
$$t^2t'=12,$$

$\therefore$ $$t=2,\quad t'=3.$$

38. *The canonizant.*—This is a name given by Dr. Sylvester to a determinant which is used in the extension of the method of the last Article. The theory assumes that a quantic of the

fifth degree can be reduced to a sum of three terms of the fifth degree, one of the seventh degree to the sum of four terms of the seventh degree, and thus for every odd degree; and then proceeds to make the assumption good in the following manner. The transformation is supposed to be effected, as before, by letting

$$s = lx + my, \quad t = l'x + m'y, \quad v = l''x + m''y.$$

The theorem then requires that

$$(a, b, c, d, e, f \between x, y)^5 = s^5 + t^5 + v^5.$$

Since the right-hand member of this equation contains implicitly as many constants as the given quantic, it must be capable of expressing that quantic when s, t, v have been properly determined.

Let $u =$ the left-hand member, and U the right-hand member of the above equation; then, by successive differentiation, we shall have

$$U \begin{vmatrix} \frac{d^4}{dx^4} & \frac{d^4}{dx^3\,dy} & \frac{d^4}{dx^2\,dy^2} \\ \frac{d^4}{dx^3\,dy} & \frac{d^4}{dx^2\,dy^2} & \frac{d^4}{dx\,dy^3} \\ \frac{d^4}{dx^2\,dy^2} & \frac{d^4}{dx\,dy^3} & \frac{d^4}{dy^4} \end{vmatrix} = U \begin{vmatrix} \frac{d^4}{dx^4} & \cdot & \cdot \\ \cdot & \frac{d^4}{dx^2\,dy^2} & \cdot \\ \cdot & \cdot & \frac{d^4}{dy^4} \end{vmatrix},$$

or the symmetrical determinants,

$$\begin{vmatrix} ax+by & bx+cy & cx+dy \\ bx+cy & cx+dy & dx+ey \\ cx+dy & dx+ey & ex+fy \end{vmatrix} = \begin{vmatrix} l^2 s & l'^2 t & l''^2 v \\ lms & l'm't & l''m''v \\ m^2 s & m'^2 t & m''^2 v \end{vmatrix}$$

$$\times \begin{vmatrix} l^2 & l'^2 & l''^2 \\ lm & l'm' & l''m'' \\ m^2 & m'^2 & m''^2 \end{vmatrix} = s\,.\,t\,.\,v \begin{vmatrix} l & l' \\ m & m' \end{vmatrix}^2 . \begin{vmatrix} l' & l'' \\ m' & m'' \end{vmatrix}^2 . \begin{vmatrix} l'' & l \\ m'' & m \end{vmatrix}^2 .$$

That is, if the expansion of

$$\begin{vmatrix} ax+by & \cdot & \cdot \\ \cdot & cx+dy & \cdot \\ \cdot & \cdot & ex+fy \end{vmatrix}$$

yields the factors s, t, v, then these factors will differ from the factors $(x+ty)$, $(x+t'y)$, $(x+t''y)$ by only numerical coefficients; and, consequently,

$$(a, b, c, d, e, f \between x, y)^5 = T(x+ty)^5 + T'(x+t'y)^5 + T''(x+t''y)^5.$$

39. In the same manner, to find the condition that a quantic of even degree can be reduced to the sum of $\frac{n}{2}$ n^{th} powers, where n is even.

The nature of this condition is seen from the last Article. The determinant formed from the n differentials will, on the supposition that the quantic can be reduced to the sum of $\frac{n}{2}$ n^{th} powers, vanish by the same process which proved that a quantic of odd degree, as for instance the fifth, could be reduced to a sum of three powers of the same degree. The determinant formed from the n differentials in the latter case being a covariant, gave the necessary factors s, t, v, while, in the case now under consideration, the proposed determinant, it will be seen, gives an invariant whose vanishing proves that the quantic can be reduced to a sum of powers each of the n^{th} degree.

To see if $2x^4+12x^3+30x^2+36x+17 = S$ can be reduced to a sum of two fourth powers, we take the fourth differentials as in the last Article and we obtain the determinant

$$\begin{vmatrix} 2 & 3 & 5 \\ 3 & 5 & 9 \\ 5 & 9 & 17 \end{vmatrix} = 0.$$

The vanishing of this determinant shows that in this case the reduction is possible. To obtain the binomials, we equate like powers of S and s, v, in $S = s^4 + v^4$, and we find

$$s = x + 1, \quad v = x + 2.$$

40. It is hardly necessary to carry this proof into the higher powers. But it may be said, in general, that if the quantic does not break up into sums of powers of binomials, it will be sufficient to add to these powers some multiple of their product or product of their powers, as

$$(a,\ b,\ c,\ d,\ e \between x,\ y)^4 = s^4 + t^4 + 6Ds^2t^2,$$

and $$(a,\ b,\ c,\ d,\ e,\ f,\ g \between x,\ y)^6 = s^6 + t^6 + u^6 + Estu.$$

That these are the least or canonical forms may be seen by extending the proof already given. The subject in such form as developed by Sylvester and others would not be necessary here.

41. *Combinants.*—We have seen (Art. 20) that the eliminant of a system of linear equations is an invariant. An invariant or eliminant of a system of equations or quantics of a uniform degree higher than the first is called a *combinant.* One peculiarity of the combinant is that it satisfies the equation

$$\frac{a_1 dC}{da} + \frac{b_1 dC}{db} + \&c. = 0,$$

where C is the combinant of

$$a\,x^n + nb\,x^{n-1} + \&c. = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$a_1 x^n + nb_1 x^{n-1} + \&c. = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$\&c. \qquad \&c.$$

42. Another peculiarity to be observed is, that if a pair of quantics have a common factor, their Jacobian will contain this factor in the second degree.

Take the equations as above, and form their Jacobian, and the truth of this will be evident; or, let a be a common factor in

$$u = ax^2 + 3ay^2,$$
$$v = 2ax^2 + ay^2,$$

then

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = -20a^2xy.$$

If, in (1) and (2), (Art. 41), $n = 2$, we shall have, for J,

$$\begin{vmatrix} ax + by & bx + cy \\ a_1x + b_1y & b_1x + c_1y \end{vmatrix} = J,$$

whose discriminant we find to be

$$4\,(ab_1 - a_1b)\,(bc_1 - b_1c) - (ac_1 - a_1c)^2.$$

This, as we have seen, is the eliminant of u and v as quadratics; or, in other words, we find that in this case, at least, the J of u and v contains their eliminant as a factor: and this is a truth to be observed when $n = 3, 4$, &c., in which cases the discriminant of J, as is evident, will be composed of the eliminant of the quantics and some other factor whose form may be determined.

43. If (1) and (2) above be represented by u and v, then $u + hv$ represents a locus common to u and v; and, by assigning varying values to h, we shall obtain a system of quantics some of which will contain square factors; and in the involution of points formed by these quantics there will be as many double points as there are quantics which contain square factors. The number of these is seen to be $2\,(n-1)$, or is the same as the order in the coefficients of the discriminant. The number of double points may then be determined by the Jacobian of u and v. If $u + hv$ has a factor $(x-a)^2$, then a will satisfy $\frac{du}{dx} + h\,\frac{dv_1}{dx} = 0$, and $\frac{du}{dy} + h\,\frac{dv_1}{dy} = 0$. It will evidently satisfy the

equation obtained by eliminating h, and therefore the Jacobian of (1) and (2). We thus have an easy method of determining the number of double points resulting from the involution of these quantics. In this form we see that h can be so determined that $u+hv$ shall contain the square factor $(x-a)^2$; and, by adding another condition, we may determine the value of constants so that the quantic shall contain $(x-a)^3$. The coefficient in (1) and (2) will then be of the degree $3\,(n-2)$. Conversely, if $(x-a)^3$ exists as a factor in $u+hv+mt$, this factor in the first degree will exist in the three second differential coefficients, and consequently in their eliminant with respect to h and m; that is

$$\begin{vmatrix} \frac{d^2u}{dx^2} & \frac{d^2v}{dx^2} & \frac{d^2t}{dx^2} \\ \frac{d^2u}{dx\,dy} & \frac{d^2v}{dx\,dy} & \frac{d^2t}{dx\,dy} \\ \frac{d^2u}{dy^2} & \frac{d^2v}{dy^2} & \frac{d^2t}{dy^2} \end{vmatrix} = 0,$$

which gives the number of triple points in the above system $u+hv+mt$; and, if $y=1$, it expresses the number of these points on the axis of x, or $3\,(n-2)$, which fulfils the condition of a combinant.

44. *Tact-invariant.*—When we find the eliminant of (1) and (2), and equate it to zero, we express the condition that the two curves may be tangent to each other. If we express also the existence of a cubic factor in any quantic of the series $u+hv$, that is, a cuspidal curve, by $U=0$ and by $V=0$, one having two double points, or two square factors, and by $W=0$, what has been described above as the tact-invariant; then the discriminant of $u+hv$ with respect to h will contain U, V, W as factors.

If u and v are tangent to each other, then the discriminant of $u+hv$ will, as a function of h, have a square factor; in other words, when expressed geometrically, it is the condition that a curve has a double point.

45. The tact-invariant is of the order $3n(n-1)$ in the coefficients. Hence, if we have three surfaces L, P, Q (of l, m, n degrees), the condition that two of the lmn points of intersection will coincide is called, in this case, the tact-invariant, and the coefficients of L are in the degree $mn(2l+n+m-4)$, and so of P and Q.

The tact-invariant of two surfaces aL and P have the coefficients of L in the degree $m(l^2+2lm+3m^2-4l-8m+6)$.* These results are obtained from quantics of four variables.

The geometrical importance of these results will be further seen.

46. As to *the number of invariants of a binary quantic*, we have already seen that a quadratic has one, that a cubic has one, each of these being the discriminant of the given quantic.

If we take the next in order, the quantic

$$(a, b, c, d, e \between x, y)^4,$$

we can easily determine the number of ordinary invariants, omitting from our enumeration those which are expressible as rational and integral functions of the same or lower degrees. Remembering that the invariant must satisfy the differential equation

$$a_0 \frac{dI}{da_1} + 2a_1 \frac{dI}{da_2} + 3a_2 \frac{dI}{da_3} + \&c. = 0,$$

and that the last invariant must be of the order 2 in the coefficients, it must therefore be of the weight 4 in the coefficients, that is,

$$Aa_4a_0 + Ba_3a_1 + Ca_2a_2.$$

Operating upon this with the differential equation, we have

$$4Aa_3a_0 + 3Ba_2a_1 + Ba_0a_3 + 4Ca_1a_2,$$

* Terquem's *Annales*, Vol. XIX., and *Quarterly Journal*, Vol. I.

which, taking A as 1, gives for B, -4, and for C, 3; and we have by substitution

$$a_4a_0-4a_3a_1+3a_2a_2,$$

or
$$ae-4bd+3c^2,$$

as the invariant function, which is the same as would have resulted by actual transformation. Had we followed the latter method, we should have found that the function of the new would be equal to the old when multiplied by the fourth power of the modulus, $(lm'-l'm)^4$ or Δ^4, or, written fully,

$$AE-4BD+3C^2=\Delta^4(ae-4bd+3c^2) \;\ldots\ldots\ldots\ldots\; (1).$$

Proceeding now to the invariant of the third order in the coefficients, we see that the weight would be 6, and must be of the general form

$$Aa_0a_4a_2+Ba_1a_2a_3+Ca_0a_3a_3+Da_1a_1a_4+Ea_2a_2a_2,$$

which embraces all possible forms.

By applying the differential equations as before, we have

$$ace+2bcd-ad^2-eb^2-c^3,$$

or
$$ACE+2BCD-AD^2-EB^2-C^3$$
$$=\Delta^6(ace+2bcd-ad^2-eb^2-c^3)\ldots\ldots\ldots(2).$$

If the Δ^6 does not follow clearly by symmetry, the actual transformation will make it evident. If we proceed to the fourth order in the coefficients of another invariant, we shall find only a function of those already found, which therefore is not to be counted in the enumeration.

47. *Absolute Invariants.*—If we eliminate Δ between (1) and (2) in the above, we shall obtain what has been called an absolute invariant, that is,

$$(ACE+2BCD-AD^2-EB^2-C^3)^2(ae-4bd+3c^2)^3$$
$$=(AE-4BD+3C^2)^3(ace+2bcd-ad^2-eb^2-c^3)^2.$$

And if I and T represent the invariants (1) and (2), their ratio $I^3 : T^2$, as is seen, is unchanged by transformation.

48. As to the discriminant of the quartic which is the eliminant of its two first differentials, we shall see that we can arrive at a method of derivation by means of (1) and (2). We have only to remember that the eliminant vanishes if the differentials have a common factor, and that this factor will exist if the binary quantic contains a square factor. We have only, to arrive at this condition, to suppose the first two coefficients to vanish; the quantic then has a square factor, since it is divisible by y^2. It is clear, also, that the invariant of such a quantic must vanish. The one contains the other as a factor when the two first coefficients a and b vanish. Or we may state it thus:—The invariant is a symmetric function of the differences of the roots, and the discriminant is the product of the squares of the differences between any two roots (Art. 17); that is, the invariant, on the above supposition that the roots are equal, as expressed in the terms of the roots, must contain the difference between the roots taken two and two. Now, since the ratio of $I^3 : T^2$ is unchanged by transformation, a new invariant may be constructed from them, and we see that $I^3 - 27T^2$ will vanish when a and b are each 0; that is, I becomes $3c^2$, and T, $-c^3$, on that supposition. And since we know (Art. 15) that this form gives us the required order in the coefficients, we conclude it to be the discriminant, that is,

$$(ae-4bd+3c^2)^3 - 27\,(ace+2bcd-ad^2-eb^2-c^3)^2,$$

which, being of the form of $I^3 \pm kT^2$, is not commonly reckoned as distinct from I and T; and thus generally when, as in this case, I and T are expressible as an invariant, a function both rational and integral of I and T, such function is not counted as a new invariant. We would infer also, in the same manner, that, if I and T are invariants of the same degree, then $I \pm kT$ need not be counted.

To sum up our number of invariants thus far, we have

$ac-b^2$ the invariant of the quadratic $(a, b, c \between x, y)^2$,

which is the discriminant.

Next, $\quad a^2d^2-6abcd+4b^3d+4ac^3-3b^2c^2$

is the discriminant of the cubic $(a, b, c, d \between x, y)^3$ (Art. 5); that is, it is the eliminant of its two first differentials.

This is its only invariant (Art. 22).

And, lastly, the I and T of the quartic $(a, b, c, d, e \between x, y)^4$ just considered, which are two ordinary invariants.

49. *The Series of Covariants.*—It follows from the definitions of invariants and covariants, and may easily be verified, that every invariant of a covariant is an invariant of the original quantic, and the contrary; consequently the quadratic can have no other covariant than the quadratic itself; or we say that this fact follows immediately from the consideration that there are no differences of roots—there being in this case but one difference—and that there can be no function of the differences of the roots. But in the cubic, since a symmetric function of differences of roots, and differences between x and one or more of the roots, is a covariant, we can form a covariant distinct from the cubic. The form of this covariant,

$$(a^2d-3abc+2b^3,\ abd-2ac^2+b^2c,\\ -acd+2b^2d-bc^2,\ 3bcd-ad^2-2c^3 \between x, y)^3,$$

has been investigated in Art. 34, and the process need not be repeated here. We have also the Hessian which Dr. Salmon writes

$$H = \begin{vmatrix} a & b & c \\ b & c & d \\ y^2 & -xy & x^2 \end{vmatrix} = (ac-b^2)\,x^2+(ad-bc)\,xy+(bd-c^2)\,y^2.$$

These two covariants examined in connection with the quantic itself, which is also a covariant, show at once that the list for the cubic is complete. For we see that the coefficient of x^2 in each case is a, $ac-b^2$, $a^2d-3abc+2b^3$, which are called the leaders. Recurring now to the discussion (Art. 35), we find that whatever analytical relation exists between the leaders of covariants, that same or similar relation will hold with the covariants as a whole. This being the case, we need only operate upon these leaders in order to discover the successive covariants.

Thus H above is the Jacobian of the first covariant $(a, b, c, d \between x, y)^3$, or V, and the original quantic $(a, b, c \between x, y)^2$, say; so also the third covariant in the above series, whose leader is $a^2d-3abc+2b^3$, is the Jacobian of the above Hessian, and the original quantic, which in this case, the cubic, is V, and thus each succeeding covariant, is found by taking the Jacobian of the last covariant of the series and the original quantic, whatever that may be. For the cubic this last covariant is indicated by J.

50. The question whether any other covariants may be properly added to this list, as regards the cubic, may be examined as follows. We see that a, ac, a^2d, &c. are divisible by a. We find then what new functions, rational and integral, of these leaders may be formed which contain a. In this case, the leaders of H, J, $ac-b^2$, $a^2d-3abc+2b^3$, become, on the supposition that $a=0$, $4H'^3+J'^2=0$. It therefore contains some power of a. Performing the operation indicated by $4H^3+J^3=0$ and dividing by a^2, we obtain the discriminant of the cubic $\quad a^2d^2-6abcd-3b^2c^2+4ac^3+4db^3$.

Now it must be remembered that a covariant, as also an invariant, is by definition a function of differences of the roots, and that a covariant is known when its source or leading coefficient is known (Art. 36); hence these leaders, as well as resulting invariants, will satisfy concurrently the differential equation $V\left(a_0\dfrac{d}{da_1}+2a_1\dfrac{d}{da_2}+3a_2\dfrac{d}{da_3}+\text{\&c.}\right)=0$,
where V is any leader or invariant.

From this fact, and in conformity with the definition, we might, for the purposes of this classification, include the invariants with the covariant of a quantic. The above discriminant, then, may be classed with the coefficients of the covariants.

Regarded in this light, we shall find that a quantic of the n^{th} degree will have n covariants, including the quantic itself, so that each other covariant, multiplied by some power of the quantic, will be equal to a rational and integral function of the n covariants. Thus, at once, if we represent the discriminant (invariant) by Δ, we shall have

$$\Delta V^2 = J^2 + 4H^3,*$$

or, using the canonical forms,

$$a^2d^2(ax^3+dy^3)^2 = a^2d^2(ax^3-dy^3)^2+4(adxy)^3.$$

51. If in Δ we let $a=0$, we have left a quantity containing coefficients which cannot be eliminated by combining with $-b^2$ or $2b^3$. In other words, no new functions of $ac-b^2$, $a^2d-3abc+2b^3$ can be formed divisible by a. Hence we may say for the cubic the list is complete.

52. The covariants of the quartic are first the Hessian,† and then the Jacobian of this Hessian and the quartic itself must be taken. We find H to be

$$\{ac-b^2,\ 2(ad-bc),\ ae+2bd-3c^2,\ 2(be-cd),\ ce-d^2\between x, y\}^4.$$

The Jacobian has its first term, or leader, $a^2d-3abc+2b^3$ &c., which, by Prof. Cayley's symbolical representation (where the Hessian of every binary quantic is written $\overline{12}^2$, and the Jacobian of H and the quantic $\overline{12}^2$, $\overline{13}$), is easily distinguished, and indicates a basis of calculation.

* Prof. Cayley, "Phil. Trans.," 1854.

† Known in geometry as the Harmonic Conic.

53. We might state here more fully the principle of this symbolic representation.

In Arts. 30 and 34, it was shown that $\frac{d}{dx}, \frac{d}{dy}$, &c., regarded as operating symbols contragredient to x, y, &c., while transformed by a direct substitution x, y, &c., will be transformed by an inverse substitution, and the contrary; and that, representing $\frac{d}{dx}, \frac{d}{dy}$, &c., by α, β, &c., operating symbols could be formed which, substituted in the quartic, a covariant or invariant could be formed according as the variables were or were not removed by differentiation. We can thus form an operative symbol for a system of quartics by a system of determinants formed of α, β, &c. Thus $\alpha_1\beta_2-\alpha_2\beta_1$, represented by $\overline{12}$, is an invariant symbol of operation. If we operate on two quantics S and V, the result of the operation upon their product SV by $\overline{12}$ is the Jacobian.

If these are quadratics,

$$(a, b, c\between x, y)^2, \quad (a_1, b_1, c_1\between x_2, y_2)^2,$$

then the result of the operative symbol $\overline{12}^2$, or

$$\alpha_1^2\beta_2^2 - 2\alpha_1\beta_1\alpha_2\beta_2 + \alpha_2^2\beta_1^2,$$

on SV will be an invariant, *i.e.*, $ac_1+ca_1-2bb_1$.

In the same manner, $\overline{12}^2\ \overline{13}$ expresses the operative symbol (or its effect upon a binary quantic)

$$(\alpha_1\beta_2-\alpha_2\beta_1)^2(\alpha_1\beta_3-\beta_3\alpha_1).$$

54. We have then, as the effect of $\overline{12}$ on SV, the Jacobian

$$\frac{dS}{dx}\frac{dV}{dy}-\frac{dS}{dy}\frac{dV}{dx},$$

and the application to any two quantics may be expressed by

$$\left(\frac{dS}{dx}\frac{dV}{dy}-\frac{dS}{dy}\frac{dV}{dx}\right)^n,$$

or $\overline{12}^n$.

In the former case, the exponent of the power does not apply to S and V, but only to the symbols of differentiation. The result is, of course, the same in both cases—an invariant if $n =$ the degree of the quantic, since all the variables are removed by differentiation, or a covariant if n is less than the degree of the quantic. From this it will immediately appear that, if by this process, we wish to form the covariant of a single quantic, we have only to make $S = V$. Thus, if we desired to form the covariant of a single quantic with the symbol $\overline{12}^2$, or

$$\left(\frac{dS}{dx}\frac{dV}{dy} - \frac{dS}{dy}\frac{dV}{dx}\right)^2,$$

we have only to make $S = V$, and the latter symbol becomes

$$2\left[\frac{d^2S}{dx^2}\frac{d^2S}{dy^2} - \left(\frac{d^2S}{dx\,dy}\right)^2\right],$$

which, applied to two quadratics S and V, would in this case give $2(ac - b^2)$. Hence, in general, the quantic to be operated upon may be conceived to be the product of two or more quantics S, V, T, &c., whose variables are distinguished by subscripts, as x_1, y_1, x_2, y_2, &c., and when the differentiation is complete the variables are written solely x, y. Since $\overline{32}$ and $\overline{23}$ are clearly the same with opposite signs, as also $\overline{12}$ and $\overline{21}$, it will appear that either of these symbols with odd powers will, when applied to any single function as SV, cause it to vanish. Following this analogy, we can easily write the symbol for a system of ternary quadratics. If, for x_1, y_2, &c., we write $\frac{d}{dx_1}$, $\frac{d}{dy_2}$, &c. (in which the cogredient variables can be written as a determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \overline{123}),$$

we shall have, when the symbol $\overline{123}^2$ is applied to the ternary quadratics

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$a_1x^2+b_1y^2+\&c.$$
$$a_2x^2+b_2y^2+\&c.,$$

$$6\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 6\,(abc+2fgh-af^2-bg^2-ch^2),$$

i.e., six times the discriminant of the ternary quadratic when $a=a_1=a_2$ &c.

CHAPTER IV.

COMPUTATION AND GEOMETRICAL APPLICATION OF INVARIANTS.

55. THE attentive reader of the preceding pages will have now no great difficulty in making a variety of important applications of the Invariant Theory.

It is shown in works on the Conic Sections, that if V and V_1 represent two conics, there are three values of k for which $kV \pm V_1$ represents a pair of right lines.

We take
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

as the general homogeneous equation of the second degree in three variables; and this is intimately connected with

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \text{............} (1);$$

the latter being derived from the former by making $z=1$.

This latter *may* represent two right lines, and does in general, when its coefficients fulfil the relation

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc+2fgh-af^2-bg^2-ch^2=0,$$

which is obtained by the resolution of (1) as a quadratic; the above determinant being the condition necessary to make the quantity under the radical a perfect square. If we call

$$V_1 = a_1x^2+b_1y^2+c_1z^2+2f_1yz+2g_1zx+2h_1xy,$$

then $$\Delta_1 = a_1b_1c_1 + 2f_1g_1h_1 - a_1f_1^2 - b_1g_1^2 - c_1h_1^2.$$

It is not difficult to see that the three values of k, for which $kV \pm V_1$ represents a pair of right lines, is obtained by substituting $ka+a_1$, $kb+b_1$, &c., for a, b, c, &c., in Δ. Writing this result in full, we shall find that k^3 will have Δ for its coefficient; k^2 and k will have functions for their coefficients, which may be represented by θ and θ_1; and lastly, that Δ_1 appears as the absolute term; that is,

$$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0.$$

The value of

$$\theta = (bc-f^2)\,a_1 + (ca-g^2)\,b_1 + (ab-h^2)\,c_1$$
$$+2\,(gh-af)\,f_1 + 2\,(hf-bg)\,g_1 + 2\,(fg-ch)\,h_1 \;\ldots\; (2),$$

and $$\theta_1 = (b_1c_1 - f_1^2)\,a + \&c.,$$

the same as θ, the accents being interchanged.

Now between $$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0 \;\ldots\ldots\ldots\ldots\ldots\; (3),$$

and $$kV + V_1 = 0,$$

we may eliminate k, which gives

$$\Delta V_1^3 - \theta V_1^2 V + \theta_1 V_1 V^2 - V^3 \Delta_1 = 0,$$

denoting the three pairs of lines which join the four points of intersection of V and V_1.

56. Since any two conics have a common self-conjugate triangle, and since they may be written

$$V = ax^2 + by^2 + cz^2 = 0,$$

$$V_1 = a_1x^2 + b_1y^2 + c_1z^2 = 0,$$

(see T., Arts. 45, 47, 56,) or

$$V_1 = x^2 + y^2 + z^2 = 0,$$

where x is written for $x\sqrt{a_1}$, &c., we obtain, by Invariants, the three values for which $kV_1 + V$ represents right lines.

Then Δ reduces to abc,

$$\theta = ab + bc + ac, \quad \theta_1 = a + b + c, \quad \Delta_1 = 1;$$

or, were we to substitute $ka + a_1$, &c., in abc, we must have, for the required condition,

$$k^3 + k^2(a + b + c) + k(ab + ac + bc) + abc = 0,$$

which is satisfied by $-a$, $-b$, $-c$.

For another example, let us take the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0 = V,$$

and the circle $(x - x_1)^2 + (y - y_1)^2 - r^2 = 0 = V_1$.

In forming Δ from V, we must remember to affect the result by the negative sign, since c or the coefficient of z^2, as well as z^2 itself, is reduced to unity with the minus sign. Hence

$$\Delta = -\frac{1}{a^2 b^2}.$$

To obtain θ we must recur to the general equation of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

From which we find, by comparing the values of the coefficients with those in the preceding Article, that

$$\begin{aligned} a_1 &= 1, & a &= \frac{1}{a^2}, \\ b_1 &= 1, & & \\ g_1 &= x_1, & b &= \frac{1}{b^2}, \\ f_1 &= y_1, & & \\ c_1 &= x_1^2 + y_1^2 - r^2. & c &= -1. \end{aligned}$$

From these values we find θ to be

$$-\frac{1}{b^2} - \frac{1}{a^2} + \frac{x_1^2 + y_1^2 - r^2}{a^2 b^2},$$

* Students who are familiar with Salmon's "Conic Sections," will at once recognize these examples. It is believed that the treatment here given them will completely remove the difficulties which hitherto have been experienced by many in their solution.

or $$\frac{1}{a^2 b^2}(x_1^2 + y_1^2 - r^2 - a^2 - b^2).$$

In the same manner, by interchanging the accents, we find

$$\theta_1 = (x_1^2 + y_1^2 - r^2 - y_1^2)\frac{1}{a^2} + (x_1^2 + y_1^2 - r^2 - x_1^2)\frac{1}{b^2} + (1-0)(-1)$$
$$= \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 - r^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right),$$

and $$\Delta_1 = x_1^2 + y_1^2 - r^2 - y_1^2 - x_1^2 = -r^2;$$

from which the equation in k is formed.

If, instead of the ellipse, we had taken the circle

$$x^2 + y^2 - r^2 = 0,$$

V_1 remaining as before, accenting the r, we should have had

$$\Delta = -r^2,$$

since $$a = 1, \quad b = 1, \quad c = -r^2;$$

$$\theta = (-r^2 - 0)\,1 + (-r^2 - 0)\,1 + (1-0)(x_1^2 + y_1^2 - r_1^2)$$
$$= x_1^2 + y_1^2 - 2r^2 - r_1^2, \text{ by (2) of Art. 55};$$

$$\theta_1 = (x_1^2 + y_1^2 - r_1^2 - y_1^2)1 + (x_1^2 + y_1^2 - r_1^2 - x_1^2)1 + (1-0)(-r^2)$$
$$= x_1^2 + y_1^2 - 2r_1^2 - r^2,$$
$$\Delta_1 = -r_1^2,$$

as in the previous case.

57. Since Δ, Δ_1, θ, θ_1 are invariants of the system of conics under consideration, their computation should be carefully studied, because in solid, as we shall see, as well as in plane geometry, these functions are fundamental.

Take the parabola $y^2 = 2px$, and V_1 as before, the circle

$$(x - x_1)^2 + (y - y_1)^2 = r_1^2.$$

Here $b=1$, while p corresponds to g in the more general equation, as is evident from (1), Art. 55, the other coefficients reducing to zero.

We have then $\qquad \Delta = -bg^2 = -p^2,$

$$\theta = (0-p^2)\,1+(0+2p)\,(-x_1) = -p\,(2x_1+p),$$

$$\theta_1 = (x_1^2+y_1^2-r_1^2-x_1^2)\,1+(0+x_1)(-2p) = y_1^2-2px_1-r_1^2,$$

$$\Delta = -r_1^2, \text{ as before.}$$

If $\qquad x^2+y^2=r^2$ and $(x-x_1)^2+(y-y_1)^2=r_1^2$

represent two circles, and d the distance between their centres, we have, as before,

$$\Delta = -r^2,\quad \theta = d^2-2r^2-r_1^2,\quad \theta_1 = d^2-r^2-2r_1^2,\quad \Delta_1 = -r_1^2.$$

58. If we turn to equation (3), Art. 55, and observe its degree, and remember that two conics always intersect in four points, and that four points may be connected by six lines, viz., 12, 13, 14, 23, 24, 34, we may conclude that this equation is that of the three pairs of chords of intersection of the two conics.

An easy application of this equation is found in the problem, to find the locus of the intersection of normals to a conic from the ends of a chord which passes through a given point.

The equation of the normal to an ellipse is

$$a^2xy_1-b^2x_1y=c^2x_1y_1.$$

If we interchange the accents, the right line becomes a curve, in fact, an hyperbola $\quad a^2x_1y-b^2xy_1=c^2xy,$

expressing that the point on the normal is known, and that the point on the curve is sought; consequently, we see that the intersections of the given ellipse and the equation last written are points whose normals will pass through the given point; that is, $x_1\,y_1$.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0=V,$$

$$2\,(a^2xy_1-b^2x_1y-c^2x_1y_1)=V_1.$$

This latter equation, it is evident, should be, as has been done, multiplied by 2 in order to sustain the fixed numerical relation expressed in the corresponding coefficients of the general equation. The equation of the six chords joining the feet of normals through xy, the locus required when satisfying the given point, is readily formed by substituting the requisite invariants in equation (3), referred to above.

We have then

$$\Delta=-\frac{1}{a^2b^2},\qquad \theta=0,$$

since $$h_1=c^2,$$

$$g_1=b^2y',\text{ and } a_1=b_1=c_1=0,$$

$$f_1=-a^2x';$$

and therefore $$\theta_1=-(a^2x_1^2-c^4+b^2y_1^2),$$

$$\Delta_1=-2a^2b^2c^2x_1y_1.$$

Hence, if $\alpha\beta$ represent the given point, we have

$$\frac{8}{a^2b^2}(a^2\beta x-b^2\alpha y-c^2\alpha\beta)^3+\&c.=0,$$

an equation of the third degree, reducing to a conic when the axis is a part of the locus.

59. In the cubic for k, its values, for which $kV\pm V_1$ represents right lines, remain the same without reference to the coordinates in which V and V_1 are taken. In other words, the relation between the coefficients Δ, θ, &c., remains unaltered by a change of coordinates, and these coefficients for the new system are equal to those of the old, multiplied by the square of the modulus of transformation, or in general by some power of that modulus. (Art. 20.)

60. If 1 and 2 of the four points of intersection of two conics coincide, then 13 and 23 will coincide with 14 and 24. In this case the cubic in k will have two equal roots. Let us take the differential coefficient of this equation, and proceed as if to find their greatest common divisor. This condition may be expressed as

$$(\theta\theta_1 - 9\Delta\Delta_1)^2 - 4(\theta^2 - 3\Delta\theta_1)(\theta_1^2 - 3\Delta_1\theta) = 0.*$$

In this case the conics are said to touch each other, though it must not be supposed that there are not also two other real or imaginary points in which the conics meet. A great variety of examples will at once occur to the reader which will illustrate the foregoing. We might exhibit an application of the last example, Art. 58. Expressing that the two curves touch, we must have, since $\theta = 0$,

$$27\Delta\Delta_1^2 + 4\theta_1^3 = 0.$$

Now that this equation will apply to the finding of the evolute of the given curve—that is, the ellipse—we have only to remember that the coordinates of the centre of the osculatory circle and those of the evolute coincide, that two of the normals coincide which can be drawn through each point of the evolute; and we have

$$27a^2b^2c^4x^2y^2 + (a^2x^2 + b^2y^2 - c^4)^3 = 0$$

as the required equation.

61. Before passing to other applications, we may discuss the conditions under which Δ_1, θ, and θ_1 vanish.

If $\Delta_1 = 0$, how shall we interpret θ and θ_1? Since V_1 breaks up into two right lines when $\Delta_1 = 0$, we may represent these lines by α and β, and then instead of $V + kV_1$ we may write $V + 2k\alpha\beta$, whose discriminant may be found by substituting $h + k$ for h in Δ, from which we obtain

$$\Delta + 2k(fg - ch) - ck^2. \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

* This condition may be found by equating the discriminant of the given cubic in k (Article 55) to zero.

But when the coefficient of k vanishes, that is, when $fg = ch$, we have the condition that the pole of the axis of x in the general equation should lie on the axis of y; in other words, in this case, that the lines α and β are conjugate with respect to V.

Now the vanishing of the discriminant indicates, as we know, a double point in the curve, and hence the vanishing of (1) shows us that the point $\alpha\beta$ lies on the curve V; that is, the coefficient of k^2 vanishes when, in this case, $c = 0$; and consequently that, when $\theta_1=0$, the intersection of the two lines is on V.

More generally, the geometrical interpretation of $\theta=0$ may be shown if we take the trilinear equation (T. 47) of the general form $$ax^2+by^2+cz^2 = 0,$$ in which the triangle of reference is self-conjugate in respect to V_1. We have then

$$\theta = (bc-0)\,a_1+(ca-0)\,b_1+(ab-0)\,c_1.$$

Again, from (T. 53), we see that

$$f_1yz+g_1zx+h_1xy = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

represents a curve circumscribing the triangle of reference. Hence we say that, if V_1 has the form of (2), θ will vanish, since, in that case, $a_1=b_1=c_1=0$; that is, θ will vanish when the triangle of reference inscribed in V_1 is self-conjugate in respect to V. If we reverse this relation, taking the triangle of reference as self-conjugate in respect to V_1,

$$\theta = (bc-f^2)\,a_1+(ca-g^2)\,b_1+(ab-h^2)\,c_1 \ldots\ldots\ldots(3),$$

since in this case $\qquad f_1 = g_1 = h_1 = 0.$

We see that (3) will vanish if we impose the condition of equal roots in the general equation; that is, if $bc=f^2$, &c., which is the condition of coincident tangents, or that x as a line should touch V; that is, that the triangle should circumscribe V while self-conjugate in respect to V_1, in which case $\theta=0$.

62. Since $V = k\alpha^2$ represents a conic having double contact with V, α being the chord of contact, if now V represent the general equation in x, y, z, and $lx+my+nz$ the equation of a line in trilinear coordinates, the equation of any conic having double contact with V on the points of intersection with the given conic, can be written

$$kV+(lx+my+nz)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

and suppose it were required to so determine k that this equation may represent two right lines. In this case Δ remains unaffected, but θ evidently becomes

$$(bc-f^2)\,l^2+(ca-g^2)\,m^2+(ab-h^2)\,n^2+2\,(gh-af)\,mn$$
$$+2\,(hf-bg)\,nl+2\,(fg-ch)\,lm = 0.$$

But since, by hypothesis, V_1 breaks up into two right lines, $\Delta_1 = 0$, and θ also vanishes, since there is double contact, or the intersection of the two lines is on V; hence the cubic in k reduces to

$$\Delta k^3+\theta k^2 = 0.$$

In other words, there are two roots $= 0$, and we have

$$k\Delta + \theta = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

to determine the other. When there are two equal roots, the conics touch each other (Art. 59). Hence, finding the value of k in (2), and substituting it in (1), we have

$$\theta V = \Delta\,(lx+my+nz)^2,$$

which is the equation of the pair of tangents at the points where the conic is cut by the given line. Where $\theta = 0$, representing its new value as above, we have the condition that the line touches the conic, and the tangents coincide with the line.

63. It may be well here to remind the beginner, that by a tangent is understood, analytically, in general, a line meeting the curve in two coincident points, and that when

the curve breaks up, as we have supposed, into two right lines, the only tangent which can meet such a locus must be on the intersection of these right lines; and since a curve of the second degree may always have two tangents, both tangents must coincide with the line at the point of intersection.

We know that, when V and V_1 represent conics, $V+kV_1=0$ represents a conic passing through their points of intersection. If now the condition were sought that the line $lx+my+nz=0$ should pass through one of these points, we may equate z in V to 0 and in the equation of line $=1$; and then, substituting the value of y found from the equation of the line in $V=0$, we have a quadratic in x whose condition of equal roots we wrote in the last Article, viz.,

$$\theta=(bc-f^2)\,l^2+(ca-g^2)\,m^2+(ab-h^2)\,n^2+2\,(gh-af)\,mn$$
$$+2\,(hf-by)\,nl+2\,(fg-ch)\,lm.$$

Let this right member now be represented by Σ, the condition that the given line touches V. If in this expression we write $a+ka_1$ for a, $b+kb_1$ for b and c, we shall manifestly have the same condition for $V+kV_1$, or any conic of the system, which we had for V. Hence, multiplying out, we have, for the coefficient of k,

$$(bc_1+b_1c-2ff_1)\,l^2+(ca_1+c_1a-2gg_1)\,m^2+(ab_1+a_1b-2hh_1)\,n^2$$
$$+2\,(gh_1+g_1h-af_1-a_1f)\,mn+2\,(hf_1+h_1f-bg_1-b_1g)\,nl$$
$$+2\,(fg_1+f_1g-ch_1-c_1h)\,lm.$$

Representing this by Φ,* and the coefficient of k^2 by Σ_1, we have

$$k^2\Sigma_1+k\,\Phi+\Sigma=0.$$

The condition that this equation should have equal roots is $\Phi^2=4\Sigma\Sigma_1$; or is the condition that the given line should pass through one of the four points; and as the envelope of this

* When $\Phi=0$ we have the condition that the given line shall be cut harmonically by V and V_1. It is also to be observed that this condition is a contravariant of the system of conics V and V_1.

system is clearly only these four points, the equation last written may be regarded as the envelope of the system. It is to be remembered that we are here really discussing functions which remain unaltered by change of axis, because, if V and V_1 by transformation to a new set of co-ordinates become $\overline{V}$ and $\overline{V}_1$, then $V+kV_1$ becomes $\overline{V}+k\overline{V}_1$, k still remaining constant.

Now $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ is the determinant whose vanishing is the condition that the general equation may represent right lines. Differentiating this function with reference to each of its letters, we have the coefficients of Σ above. Also both Φ and Σ_1 are functions of Δ, in such manner as to possess the character of invariance.

64. If we seek the condition that

$$l\alpha+m\beta+n\gamma=0$$

shall touch the conic represented by the general trilinear equation (T. 43)

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0 \ldots\ldots\ldots(1),$$

we shall have the condition represented by Σ, as above. For the coefficients there given, $bc-f^2$, &c., we may write A, B, &c., or

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \ \ldots\ldots(2),$$

which is sometimes called the tangential equation of the conic. If between this equation and the equation of the line we eliminate n, we shall have a quadratic in $\frac{l^2}{m^2}$, and the condition of two equal roots, or that it breaks up into straight lines; or, which in this case is the same thing, the envelope of the line is

$$(BC-F^2)\,\alpha^2+(CA-G^2)\,\beta^2+\&c.=0 \ \ldots\ldots\ldots(3),$$

an equation symmetrical with Σ,—the latter in l, m, n and its coefficients in small letters, the former in α, β, γ and its coefficients in large letters. We may see, then, that the envelope of a line whose coefficients fulfil the condition Σ is the conic (1), for we have only to substitute for A, B, &c., their values $bc-f^2$, &c., and (2) becomes $\Delta V=0$ when $V=(1)$. Consequently, if we write the trilinear equation corresponding to

$$\Sigma+k\Sigma_1=0,$$

we have $$\Delta V+kD+k^2\Delta_1 V_1=0 \qquad\qquad (4),$$

in which D is symmetrical with Φ; that is,

$$D=(BC_1+B_1C-2FF_1)\,\alpha^2+\&c.,$$

an equation in x, y, z when V and V_1 have the meaning we have heretofore assigned them.

The envelope of the system (3) is

$$D^2=4\Delta\Delta_1 VV_1;$$

but the envelope in this case is the four common tangents. Hence this is the equation of the four common tangents to the two conics.

To illustrate this, take the two conics

$$x^2+3y^2+5z^2=0,$$

$$2x^2+4y^2+6z^2=0.$$

$$\Delta=15,\ \Delta_1=48,\ A=15,\ B=5,\ C=3,$$

$$A_1=24,\ B_1=12,\ C_1=8;$$

$$D=2\,(18+20)\,x^2+12\,(10+6)\,y^2+30\,(4+6)\,z^2.$$

Hence

$$(76x^2+192y^2+300z^2)^2=2880\,(x^2+3y^2+5z^2)\,(2x^2+4y^2+6z^2)$$

is the equation of the four common tangents to the two conics.

If $$3y^2-2x^2-4xy=0=V,$$

and $$\frac{x^2}{4}+\frac{y^2}{3}-1=0=V_1,$$

what is D?

65. As has been before intimated, an invariant is a function whose vanishing indicates some property of the curve independent of the axis to which it is referred. In the same manner, as we know, covariants are particular loci whose relation to the equations whence they were derived is independent of the axes of these given equations. In other words, the two functions agree so far as axes are concerned.

Turning our attention now to covariants, which, as we have seen, contain the given variables, we may find our illustration in the system of conics we have been considering, V and V_1, which we will again refer (Article 56) to their self-conjugate triangle, that is,

$$V = ax^2 + by^2 + cz^2,$$

$$V_1 = x^2 + y^2 + z^2.$$

If we proceed now as in the last Article, we find

$$A = bc, \quad B = ca, \quad C = ab,$$

$$A_1 = B_1 = C_1 = 1;$$

consequently

$$D = (ab + bc)x^2 + (bc + ba)y^2 + (ac + cb)z^2 \quad \ldots\ldots(1);$$

equation (2) of the preceding Article becomes

$$Al^2 + Bm^2 + Cn^2 = 0,$$

or the condition that a line should touch V. Hence the locus of the poles with regard to V_1 of the tangents to V is

$$Ax^2 + By^2 + Cz^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Adding (1) and (2), we have

$$(A + B + C)(x^2 + y^2 + z^2) = D.$$

Or, since (Art. 56) $\quad \theta = A + B + C,$

we have $\theta V_1 = D$ as the equation of the polar conic of V with respect to V_1 in terms of the conics of the system and the conic D. The locus $\theta V_1 = D$ is therefore a covariant of V and V_1, and this relation will not be altered when V and V_1 are transformed to othes axes. Similarly, $\theta_1 V = D$ is a locus, a covariant, the polar conic of V_1 in regard to V.

Returning to $\Phi = 0$ (see note, Art. 63), we see that it becomes, retaining the same expressions for V and V_1,

$$(b+c)l^2 + (c+a)m^2 + (a+b)n^2 = 0,$$

which may be called the tangential equation of the conic enveloped by a line cut harmonically by V and V_1. Now the trilinear equation, as found from this, is of the form of equation (3) of the last Article, that is,

$$BCx^2 + CAy^2 + ABz^2 = 0,$$

or

$$(c+a)(a+b)x^2 + (a+b)(b+c)y^2 + (c+a)(b+c)z^2 = 0 \ldots (1),$$

since in this case $A = (b+c)$, &c.

Adding the value of D to (1), and reducing, we have

$$\theta V_1 + \theta_1 V - D = 0$$

as the equation, a locus, covariant with V and V_1, expressing in terms of these conics a conic enveloped by a line cut harmonically by the conics in question. If D breaks up into two right lines, we have simply $\Delta = 0$ in equation (1),

or $$(ab+ac)(bc+ba)(ac+ab) = 0.$$

66. It would be a profitable exercise for the reader, at this stage, to reduce a few conics to the forms

$$x^2 + y^2 + z^2 = 0,$$
$$ax^2 + by^2 + cz^2 = 0.$$

This can be done with the help of

$$\Delta k^3 + \theta k^2 + \theta_1 k + \Delta_1 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

That is, the roots of this equation will give us the new a, b, c; then we shall have

$$x^2 + y^2 + z^2 = V, \quad ax^2 + by^2 + cz^2 = V_1,$$

when V and V_1 are the given conics.

We shall still need one more equation, and for this we can conveniently use equation (1) of the preceding Article,

$$(ab+ac)x^2+(bc+ab)y^2+(ac+cb)z^2 = D,$$

with this caution, that, as the discriminant of V is 1, D must be divided by Δ to put the three equations upon the same relation.

Thus, if V and V_1 are

$$x^2-2xy+2y^2-4x+6y=0,$$

$$3x^2-6xy+5y^2-2x-1=0,$$

we see these are of the general form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The Δ of the first is -5 (Art. 55),

$$\theta=-14,\ \theta_1=-9,\ \Delta_1=-11;$$

and since (Art. 56) the roots of (1), when the conics are referred to their self-conjugate triangle, are $-a$, $-b$, $-c$, the actual form of (1) for numerical use must be

$$\Delta k^3-\theta k^2+\theta_1 k-\Delta_1=0,$$

or in this case

$$-5k^3+14k^2-9k+11=0 \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In order to calculate the covariant D, we must first know A, B, C, A_1, B_1, C_1, &c.

These may be computed by equation (2), Art. 64.

V gives us	V_1 gives
$a=1,$	$a_1=3,$
$h=-1,$	$h_1=-3,$
$b=2,$	$b_1=5,$
$g=-2,$	$g_1=-1,$
$f=3,$	$f_1=0,$
$c=0;$	$c_1=-1.$

As given in Art. 63,

$$A=bc-f^2,\quad B=ca-g^2,\quad C=ab-h^2,$$

$$F=gh-af,\quad G=hf-bg,\quad H=fg-ch.$$

In the same manner $A_1 = b_1c_1 - f_1^2$, &c.

The value of D must be computed from the general equation, which we now write in full,

$$(BC_1+B_1C-2FF_1)x^2+(CA_1+C_1A-2GG_1)y^2$$
$$+(AB_1+A_1B-2HH_1)z^2+2(GH_1+G_1H-AF_1-A_1F)yz$$
$$+2(HF_1+H_1F-BG_1-B_1G)xz$$
$$+2(FG_1+F_1G-CH_1-C_1H)=D.$$

Now suppose the roots of (1) to be represented by a, b, c (the new a, b, c), and we shall have

$$X^2+Y^2+Z^2=V,$$
$$aX^2+bY^2+cZ^2=V_1,$$
$$(ab+ac)X^2+(bc+ab)Y^2+(ac+ab)\ Z^2=\frac{D}{\Delta}.$$

From which we can obtain the values of X, Y, Z, which were required. The reader can complete this example. These computations are important on account of their frequent occurrence in geometrical investigations, as will be seen in a succeeding Tract.

67. Another of a large class of examples will show how invariants determine the situation of a conic, as for example a fixed locus.

Let us take V, a curve circumscribing the triangle of reference (T., Art. 53),

that is, $$2(u\beta\gamma+v\gamma\alpha+w\alpha\beta)=0.$$

Let V_1 be touched by two sides of the triangle. This can be represented by the tangential equation, in this case (T., Art. 54), by $$\alpha^2+\beta^2+\gamma^2-2\beta\gamma-2\gamma\alpha-2\alpha\beta\,(1+wk),$$

since $\alpha=0$, $\beta=0$, in each case, satisfies the equation, giving perfect squares. Then will $kV+V_1$, a conic passing through their intersections, be touched by the third side of the triangle. Computing the invariants as before, we have

$$\Delta = 2uvw,$$

$$\theta = -u^2 - v^2 - w^2 - 2vw - 2uw - 2uv - 2uvwk$$

$$= -(u+v+w)^2 - 2uvwk,$$

$$\theta_1 = 2(u+v+w)(2+wk), \quad \Delta_1 = -(2+hk)^2.$$

From which we obtain

$$\theta_1^2 = 4\Delta\Delta_1 k + 4\theta\Delta,$$

and, eliminating the parameter k between this last equation and $kV+V_1$, we have the envelope of the third side of the triangle of reference; or—which, in this case, is the same thing—by substituting the value of k, derived from that equation, in the latter, we obtain, plainly, a fixed conic touched by the third side, that is,

$$(\theta_1^2 - 4\theta\Delta)\, V = 4\Delta\Delta_1 V_1.$$

When $\theta_1^2 = 4\theta\Delta$, $k = 0$, and is simply the condition that the three sides of the triangle are touched by V_1.

68. If l and m are any lines at right angles to each other through a focus, we can construct an equation, a particular form of

$$u^2\alpha^2 + v^2\beta^2 = w^2\gamma^2, \qquad \text{(T., Art. 47)}$$

that is,

$$l^2 + m^2 = e^2\gamma^2,$$

where γ, the polar of the focus, is the directrix. If $e = 0$, as in the circle, we have the equation which determines the direction of the points at infinity on any circle; or, in other words,

$$l^2 + m^2 = 0$$

is the tangential equation of these points, or the condition that the line

$$lx + my + n = 0$$

should pass through one of them.

Now the necessary relation between these constants, in order that this line may touch the curve represented by the general equation, sometimes called the tangential equation of the curve, is given Art. 63, equation (2). Distinguishing this by S, let us proceed to examine the discriminant formed from

$$S+k(l^2+m^2),$$

which is $$\Delta^2+k\Delta(a+b)+k^2(ab-h^2).$$

Form also the discriminant of

$$S+kS_1,$$

which is $$\Delta^2+k\Delta\theta_1+k^2\Delta_1\theta+k^3\Delta_1^2,$$

and we see that $a+b$ corresponds to θ_1 and $ab-h^2$ to θ. Hence we say that, the invariants of any conic and a pair of points at infinity being formed, we can express the condition, by placing $\theta_1 = 0$, that the curve is an equilateral hyperbola, and by $\theta = 0$, that it is a parabola. This result follows from the theory of invariants,—viz., that whatever homogeneous relation is seen to exist in the one case will also exist in the other, irrespective of the coordinates in which the curves are expressed or the axes to which they are referred.

We now seek for the corresponding expression in Trilinear Coordinates. The length of the perpendicular on one of these four imaginary common tangents from any point must be infinite. Hence the denominator of p (T., 20) must be put $= 0$, that is,

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C = 0$$

must be the general tangential equation of the points in question in trilinear coordinates. Combining this with S, as before, we find that θ_1 corresponds to

$$a+b+c-2f\cos A-2g\cos B-2h\cos C,$$

which, equated to 0, is the condition that the conic $S+kS_1$ shall represent an equilateral hyperbola.

In this computation the coefficient of k only, it is evident, need be formed, which divided by Δ must give the condition sought. To find the condition that the curve shall represent a parabola, it will be necessary to form the coefficient of k^2 and then divide this result by Δ_1.

68. By the theory of foci, the four tangents drawn through the two imaginary points at infinity on any circle form a quadrilateral, in which two of these vertices are real and the foci of the conic. Now, since $S+kS_1$ touches the four tangents common to S and S_1, it will represent these two vertices or foci in question, when k has been so determined that the conic $(S+kS_1)$ reduces to a pair of points, with the condition that S_1 represents the two points at infinity.

To find these foci, we proceed to find the value of k in

$$(ab-h^2)\,k^2+\Delta\,(a+b)\,k+\Delta^2=0,$$

which, substituted in $S+k\,(l^2+m^2)$, gives two factors, viz.,

$$\left(l\frac{x_1}{z_1}+m\frac{y_1}{z_1}+n\right)\left(l\frac{x_2}{z_2}+m\frac{y_2}{z_2}+n\right),$$

in which $\dfrac{x_1}{z_1}, \dfrac{y_1}{z_1}$ and $\dfrac{x_2}{z_2}, \dfrac{y_2}{z_2}$ are the coordinates of the foci, one value of k giving the real and the other the imaginary foci.

As a simple illustration, let us seek the coordinates of the focus of

$$x^2+2xy+y^2-2x-2y+2=0.$$

Here $ab-h^2=0$, and consequently

$$\Delta^2+k\Delta\,(a+b)+k^2\,(ab-h^2)$$

reduces to

$$2k\Delta+\Delta^2=0.$$

But

$$\Delta=2+2-1-1-2=0.$$

Hence S, or

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm+k\,(l^2+m^2),$$

reduces to $\quad l^2+m^2-2lm \quad$ or $\quad (l-m)\,(l-m).$

Therefore the cordinates of the focus are 1, 1, if we regard z_1 as the linear unit in the equation of the line

$$lx_1 + my_1 + nz_1.$$

But if these variables are conceived of as functions of one another, or the line as a function of the variables, then, as $z_1 = 0$ and the coordinates are represented by $\frac{x_1}{z_1}, \frac{y_1}{z_1}$, these become infinite, which result is still consistent with the geometrical conception of the foci of the parabola, where one focus is regarded as at infinity.

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