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MATHEMATICAL TRACTS.

No. II.

TRILINEAR COORDINATES.

TRACTS

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 2.

TRILINEAR COORDINATES.

BY

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"Ἐθέλω σοι εἰπεῖν ὥσπερ οἱ γεωμέτραι."
PLATO: GORGIAS.

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MY acknowledgments are due to R. TUCKER, Esq., M.A., Honorary Secretary of the London Mathematical Society, for valuable assistance rendered in passing these sheets through the press.—W. J. W.

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PREFACE TO TRACT NO. II.

MINISTERIAL and other duties have prevented the earlier appearance of this Tract. The delay has afforded an opportunity to those persons who have become acquainted with the proposed plan of this Series of expressing their opinion upon the merits of such an undertaking.

A considerable number of Professors and Amateurs have been pleased to signify their approval of this effort, and to give me more than deserved commendations. I have no object in referring to this, except so far as to certify that the purpose in view is a good one, and that the means adopted, while novel, are likely to prove in a fair measure successful. I take this opportunity of again urging upon those to whom these Tracts may come the great importance of the study of the Modern Mathematics, not only in their various subjects as educative instruments, but also as the best media of investigation. The extent and value of the new methods, together with the duties of those capable of teaching them, are happily expressed in a letter to me from M. Hermite, dated Paris, October 28, 1876, who will probably pardon the liberty I take with his communication, on the ground that the following extract is of public importance:—

"Les vues exposées par vous, Monsieur, dans la préface de cet ouvrage [Tract No. I.] sur les obligations qu'imposent à l'enseignement les grands progrès de la science de notre temps, je les adopte pleinement, et, autant qu'il m'a été possible, j'ai essayé de m'y conformer dans mon Cours d'Analyse de l'Ecole Polytechnique. Une grande transformation s'est déjà faite et continue encore de se faire dans le domaine de l'Analyse ; des voies nouvelles plus fécondes et je crois aussi plus faciles ont été ouvertes, et c'est l'œuvre de ceux qui veulent servir la science et leur pays de discerner ce que les éléments peuvent recevoir de l'immense élaboration qui s'est accomplie depuis Gauss jusqu'à Riemann."

I am also indebted to Prof. Benj. Peirce, of Harvard, for a communication in reference to the form of Laplace's equation for secular perturbations, referred to on p. 41 of Tract No. I.

Without detracting from the value of the Ancient Geometry, it is believed that a considerable portion might be omitted, if necessary, to give place to the Modern, and that our regular college curriculum would be greatly enriched by such substitution.

In any event, I hold it to be the duty of every teacher of Geometry, whether in the form of analysis or synthesis, to incorporate in his instructions large masses of the New Geometry, unless, indeed, there happens to be a Chair devoted to this especial science.

In presenting Trilinear Coordinates, it is not proposed to supersede the Cartesian, nor even to regard them as inseparable from them ; but to show (as Dr. Salmon has shown) the peculiar province and power of each. In this Tract it has not been thought necessary to advance far in this comparison. The student will quickly see where he can most advantageously employ the one or the other, or, leaving both, press into his service the Triangular or Tangential Coordinates.

All that could be attempted in a work of this size is to give a syllabus of the more common equational forms, and to exhibit, in as simple a manner as possible, their genesis.

There are other systems of Coordinates which space did not allow me to exhibit; the *quadrilinear*, which involves four straight lines as lines of reference, is one of some importance.

Another form of Coordinates I will just mention, the description of which has been communicated to me by Rev. Thos. Hill, D.D., LL.D., late President of Harvard. These Coordinates consist in defining a curve by expressing the length of a perpendicular let fall from the origin upon a normal as the function of its direction. Thus, if θ represent the angle contained by the perpendicular and the axis of X, then $p = f(\theta)$. These are known in this form as Watson's Coordinates. Dr. Hill has modified this system, and succeeded in achieving some very interesting results. (See Proceedings of the American Association for the Advancement of Science, 1873—75.)

For my first interest in the subject of this Tract I am indebted to a paper read before the Royal Society of Edinburgh in 1865, and published in the *Messenger of Mathematics* of the year following, the author of which, Rev. Hugh Martin, D.D., has exhibited in that paper much of the power and originality which characterise his well-known treatise upon "The Atonement."

It may be said, however, that works upon Modern Geometry do in general suggest the treatment of their subjects by the method of Trilinear Coordinates. They do, indeed, suggest far more than has been attempted here. In the works of Mulcahy, Townsend, Salmon, Ferrers, Whitworth, the recent volumes of Dr. Booth, Carnot, Steiner, Serret, Rouché and Comberousse, Bobillier, Cremona, Briot and Bouquet, Chasles, together with the journals *Annali di Matematica pura ed applicata* (of which Cremona is co-editor), *Comptes Rendus des Séances,* that of Crelle and Borchardt, *Nouvelles Annales de Mathématiques,* may be found much that leads to, and much more that leads beyond, that which now follows.

Books, at best, are but poor substitutes for the living teacher. Under familiar, oral teaching the difficulties which otherwise too frequently envelope the student rapidly disappear. Hence I would again emphasize the importance of admitting these subjects to our colleges as parts of the regular course.

Since the publication of Tract No. I., the heads of two of our leading Universities have made haste to inform me that some parts of the Modern Mathematics I am endeavouring to enforce and popularise are taught in their colleges. I profoundly wish that these exceptions were made the rule.

W. J. W.

CHAMBERSBURG, PA.; 1877.

TRILINEAR COORDINATES.

CHAPTER I.

FUNDAMENTAL EQUATIONS.

1. The fundamental equation of the straight line in Trilinear Coordinates is

$$l\alpha + m\beta + n\gamma = 0.$$

2. The apparatus for expressing this conception consists of a triangle of reference, whose sides are called the three lines of reference.

3. The angular points of this triangle are indicated by A at the vertex, B at the left, and C at the right; the lengths of the sides opposite these angles by a, b, c; and the perpendicular distances of any point from BC, CA, AB by α, β, γ.

The distance α may be described as reckoned downward or upward from the given point, β to the right, and γ to the left.

4. We may say, in general, that the position of a point in a plane is known implicitly when its perpendicular distances from any two sides of the proposed triangle are given. Its perpendicular distance from the third side is then given by these data, for manifestly

$$\frac{2\Delta - (\beta b + c\gamma)}{a} = \alpha,$$

where $\Delta =$ area of given triangle.

5. By attention to the figure, which scarcely need be drawn, we are clearly presented with the equation

$$a\alpha+b\beta+c\gamma=2\Delta \text{ ....}(1),$$

which is found by taking the sum of the areas of the three triangles APC, APB, BPC, $\frac{b\beta}{2}$, $\frac{c\gamma}{2}$, $\frac{a\alpha}{2}$ respectively.

Observing that $\frac{a}{2}=r\sin A$, $\frac{b}{2}=r\sin B$, $\frac{c}{2}=r\sin C$, (1) may be written

$$\alpha\sin A+\beta\sin B+\gamma\sin C=\frac{\Delta}{r}=\text{V},$$

where r = radius of the circumscribing circle. These equations hold, whether the point is situated below BC, within the triangle, or above the vertex.

In the first case, by convention, $a\alpha$ is regarded as negative; in the second, each term as positive; while in the last $a\alpha$ is alone positive.

6. It will be observed also that the point is equally determined if the ratios of the three perpendiculars are given, for we see at once that each ratio determines a locus, which is a line drawn through the angle upon which the point is situated. The point sought is at the intersection of these lines.

7. Before proceeding further, it may be well to exhibit in full the process for deriving the equation of the straight line (Art. 1).

Let P_1, P_2 be the given points; $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ their coordinates; and P_1P_2 the straight line whose equation is to be determined. Take any point P on this line, and let its coordinates be α, β, γ; then, by similar triangles,

$$PP_1 : PP_2 :: \alpha_1-\alpha : \alpha-\alpha_2 : \beta_1-\beta : \beta-\beta_2 : \gamma_1-\gamma : \gamma-\gamma_2;$$

or, taking the last two ratios, we are immediately presented with the two determinants (D. 2; *i.e.*, Tract No. I., Art. 2)

$$\begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} = \begin{vmatrix} \gamma & \beta \\ \gamma_1-\gamma_2 & \beta_1-\beta_2 \end{vmatrix};$$

likewise
$$\begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix} = \begin{vmatrix} \alpha & \gamma \\ \alpha_1-\alpha_2 & \gamma_1-\gamma_2 \end{vmatrix},$$

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} \beta & \alpha \\ \beta_1-\beta_2 & \alpha_1-\alpha_2 \end{vmatrix}.$$

If now we multiply these equations respectively by α, β, γ, and add, we shall have at once

$$\alpha \begin{vmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{vmatrix} + \beta \begin{vmatrix} \gamma_1 & \alpha_1 \\ \gamma_2 & \alpha_2 \end{vmatrix} + \gamma \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = 0 \;\ldots\ldots\; (1).$$

Let now these determinants in this last result be represented by l, m, n respectively, and we have

$$l\alpha + m\beta + n\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

And since l, m, n represent constants, and since also α, β, γ are the coordinates of any point of the line, this equation expresses, as before stated in (Art. 1), the conception of the general equation of the straight line in trilinear coordinates.

Cor. 1.—This is also plainly the equation of a straight line through two given points.

Cor. 2.—The ratios represented by l, m, n are manifestly constant whatever the position of P on the locus, which involves also the deduction that this locus must be a straight line.

8. *The condition of concurrence.*—Let the straight lines be

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2).$$

These equations, regarded as simultaneous, must have α, β, γ as the coordinates of a common point. To obtain the ratios, we are presented with the determinants (D., Arts. 10, 12)

$$\begin{vmatrix} 1 & 1 & 1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} + \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} + \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix};$$

otherwise

$$\alpha : \beta : \gamma :: \begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ n_2 & l_2 \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}.$$

Hence the trilinear ratios of the point of intersection are determined.

COR.—The general equation of a straight line passing through their point of intersection may be written

$$l\alpha + m\beta + n\gamma = k(l_1\alpha + m_1\beta + n_1\gamma) \quad \text{...........(3)},$$

where k is any constant; for the locus of (3) must pass through every point common to the loci of (1) and (2).

9. Three straight lines, as

$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$
$$l_2\alpha + m_2\beta + n_2\gamma = 0,$$
$$l_3\alpha + m_3\beta + n_3\gamma = 0,$$

present the determinant (D. 6)

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

as the condition that three straight lines shall have a point in common.

10. *The condition of parallelism.*—Let the two straight lines be

$$l\,\alpha + m\,\beta + n\,\gamma = 0,$$
$$l_1\alpha + m_1\beta + n_1\gamma = 0.$$

Let us find the condition of parallelism.

Suppose $\alpha, \beta, \gamma, f, g, h$ the coordinates of two points in the former; $\alpha_1, \beta_1, \gamma_1, f_1, g_1, h_1$ the coordinates of any two points in the latter.

If these lines are parallel, the geometry of the figure requires

$$\alpha - f : \beta - g : \gamma - h :: \alpha_1 - f_1 : \beta_1 - g_1 : \gamma_1 - h_1.$$

Let us seek an expression for this in terms of the constants of the given equations and the triangle of reference.

Remembering (Art. 5) that

$$a\alpha+b\beta+c\gamma = 2\Delta,$$

and consequently $\quad af+bg+ch = 2\Delta,$

we have $\quad a(\alpha-f)+b(\beta-g)+c(\gamma-h) = 0 \;\ldots\ldots\ldots\ldots(1).$

Also, since $\quad l\alpha+m\beta+n\gamma = 0,$

and $\quad lf+mg+nh = 0,$

we obtain $\quad l(\alpha-f)+m(\beta-g)+n(\gamma-h) \;\ldots\ldots\ldots\ldots\ldots(2).$

Equations (1) and (2) give the eliminant (D. 39)

$$\begin{vmatrix} 1 & 1 & 1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0,$$

from which we derive the ratios

$$\alpha-f : \beta-g : \gamma-h :: \begin{vmatrix} m & n \\ b & c \end{vmatrix} : \begin{vmatrix} n & l \\ c & a \end{vmatrix} : \begin{vmatrix} l & m \\ a & b \end{vmatrix},$$

likewise

$$\alpha_1-f_1 : \beta_1-g_1 : \gamma_1-h_1 :: \begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix};$$

hence

$$\begin{vmatrix} m & n \\ b & c \end{vmatrix} : \begin{vmatrix} n & l \\ c & a \end{vmatrix} : \begin{vmatrix} l & m \\ a & b \end{vmatrix} :: \begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} : \begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} : \begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix}.$$

Multiplying each of these ratios by l_1, m_1, n_1, and remembering how they were derived, we have, by restoring,

$$l_1\begin{vmatrix} m & n \\ b & c \end{vmatrix} + m_1\begin{vmatrix} n & l \\ c & a \end{vmatrix} + n_1\begin{vmatrix} l & m \\ a & b \end{vmatrix}$$

$$:: l_1\begin{vmatrix} m_1 & n_1 \\ b & c \end{vmatrix} + m_1\begin{vmatrix} n_1 & l_1 \\ c & a \end{vmatrix} + n_1\begin{vmatrix} l_1 & m_1 \\ a & b \end{vmatrix}.$$

These are (D. 6) the expanded determinants for

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} :: \begin{vmatrix} l_1 & m_1 & n_1 \\ l_1 & m_1 & n_1 \\ a & b & c \end{vmatrix}.$$

By (D. 7) the right-hand determinant vanishes, and hence

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0\ *$$

is the condition of parallelism; or, by reverting this determinant, it can be written (D. 12)

$$aA + bB + cC = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

and in this form is easily remembered.

11. *Excursus on the straight line.*—We have obtained (Art. 5) the equation

$$\alpha \sin A + \beta \sin B + \gamma \sin C = \frac{\Delta}{r} = \text{a constant};$$

and therefore we may write

$$l\alpha + m\beta + n\gamma + k(\alpha \sin A + \beta \sin B + \gamma \sin C) = 0$$

as the parallel of the line

$$l\alpha + m\beta + n\gamma.$$

This follows from the analogy of the Cartesian coordinates, where, it will be remembered, two lines differing by only a constant are parallel. Also, if two equations are so connected that their difference is ever a constant, their sum represents their parallel and is situated half-way between them.

In the last Art., equation (3) is the result, in fact, of elimination between three equations, one of which is the impossible equation $$a\alpha + b\beta + c\gamma = 0;$$

impossible at least in any finite conception, since we have

* That this determinant may rigorously be equated to 0 is evident from the consideration of the ratios, when it will be seen we have been, in fact, concerned with only *one* equation.

proved it equal, in every position of the origin, to the area of the triangle of reference. Here again, after the analogy of the Cartesian, of which trilinear coordinates may be regarded as a particular case,* we may interpret

$$a\alpha + b\beta + c\gamma = 0$$

as a line situated at an infinite distance from the origin, or we may say that every straight line may be regarded as parallel to the straight line at infinity.

Thus, analytically:

The ratios (Art. 8) $\alpha : \beta : \gamma$ express the relations of the coordinates of the point of intersection of two straight lines. The *actual* values are evidently given by the three equations,

$$\begin{aligned} a\alpha + b\beta + c\gamma &= 2\Delta, \\ l_1\alpha + m_1\beta + n_1\gamma &= 0, \\ l_2\alpha + m_2\beta + n_2\gamma &= 0, \end{aligned}$$

where

$$\frac{\alpha}{\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}} = \frac{2\Delta}{\begin{vmatrix} a & b & c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}.$$

Writing A for the minor in the one case, and A_1 for the determinant in the latter, we have

$$\alpha = \frac{2\Delta A}{A_1}.$$

When α becomes infinite, A_1 becomes zero. But this expresses the condition of the straight line at infinity; that is, the point of intersection lies at an infinite distance.

But this is also the condition of parallelism of two straight lines.

* Salmon's Conics, p. 64.

The determinant, therefore, to represent parallel straight lines, may be written

$$A_1 = \begin{vmatrix} a & b & c \\ a & b & c \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

which identically vanishes, and where it will be seen the ratios $l : m : n$ are merged in, and have become identical with, $a : b : c$.

12. *The condition of collinearity.*—Let the three points $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, $\alpha_3\beta_3\gamma_3$ be determined in the same straight line. We see it is only necessary to accent the α, β, γ of equation (1), (Art. 7), change α_1 to α_2, β_1 to β_2, &c., and we can write the condition at once

$$\alpha_1 \begin{vmatrix} \beta_2 & \gamma_2 \\ \beta_3 & \gamma_3 \end{vmatrix} + \beta_1 \begin{vmatrix} \gamma_2 & \alpha_2 \\ \gamma_3 & \alpha_3 \end{vmatrix} + \gamma_1 \begin{vmatrix} \alpha_2 & \beta_2 \\ \alpha_3 & \beta_3 \end{vmatrix} = 0.$$

By (D. 6),

$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0,$$

which is the condition determining three points in a straight line.

The following well-known theorem will illustrate this :—

Let P be a point within the triangle of reference. Through this point let straight lines be drawn from A, B, C to meet the opposite sides respectively in A_1, B_1, C_1; these are the angular points of a triangle whose sides, when produced, will meet the corresponding sides of the first triangle in three points which lie in a straight line.

Suppose f, g, h the coordinates of the point P; α, β, γ those of any point, as A_1. Then $\alpha = 0$; and, by similar triangles, g and h will be the ratios.

A_1 will therefore be represented by $0, g, h$.

For B_1, g of course is 0, f and h its ratios, since $\beta : \gamma :: g : h$. Hence, in the same manner, B_1 is represented by

$$f, \ 0, \ h.$$

Hence the line joining A_1, B_1 is (Art. 7, Cor. 1)

$$\alpha \begin{vmatrix} g & h \\ 0 & h \end{vmatrix} + \beta \begin{vmatrix} h & 0 \\ h & f \end{vmatrix} + \gamma \begin{vmatrix} 0 & g \\ f & 0 \end{vmatrix} = 0.$$

Otherwise $$\alpha gh + \beta hf - \gamma fg = 0 \quad\ldots\ldots\ldots\ldots (1),$$

which may be written, the line

$$gh, \quad hf, \quad -fg,$$

using the coefficients only to represent the line.

Recurring again to equation (1), (Art. 7), we see that, if γ_1, γ_2 are each $=0$, we must have, for (2) of the same Article,

$$0\alpha + 0\beta + n\gamma = 0 \quad\ldots\ldots\ldots\ldots (2).$$

But this condition attaches to the line AB, which is therefore represented by $\quad 0, \ 0, \ 1.$

The intersection of A_1B_1 and AB is therefore the concurrence of (1) and (2), which (Art. 8) is the point

$$fh, \quad -gh, \quad 0;$$

or, by ratios, $$f, \quad -g, \quad 0.$$

BC and B_1C_1 will intersect, similarly, in

$$0, \quad g, \quad -h;$$

and AC, A_1C_1 in $$-f, \quad 0, \quad h.$$

Hence, since

$$\begin{vmatrix} f & -g & 0 \\ -f & 0 & h \\ 0 & g & -h \end{vmatrix} = 0,$$

the lines (AB, A_1B_1), (BC, B_1C_1), (AC, A_1C_1) meet in points which are collinear.

13. Another illustration of the use of these coordinates is found in the proof that the straight line joining the middle points of two sides of the triangle of reference is parallel to the third side. If the points be taken on BC and AC, then

equations (1) and (2) of the last Article will represent the lines to be considered, remembering only to accent two of the coordinates, when (1) becomes

$$gh_1, \quad hf_1, \quad -f_1 g\,;$$

and (2)

$$0 \quad 0 \quad 1.$$

Substituting these in the determinant of parallelism (Art. 10), we find the required expression

$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & 0 & 1 \end{vmatrix} = am - bl = ahf_1 - bgh_1,$$

by giving m and l their values; and since, if the given lines are parallel, $h = h_1$, we may write

$$af_1 = bg,$$

which accords with the geometry of the figure.

14. *The condition of perpendicularity.*—The more common method of determining this condition is by establishing, in the first place, the angular relation of a given straight line to two of the sides of the triangle of reference. For this purpose the internal bisector of one of the angles may be used as an axis. A line drawn through the vertex A, for instance, may be regarded as known when its inclination to the bisector of this angle is determined. The equation of such a line evidently is concerned with but the two coordinates β, γ.

Two lines thus drawn may be represented by

$$t\beta = s\gamma \quad \text{.........................} \quad (1),$$

and

$$t_1\beta = s_1\gamma \quad \text{.........................} \quad (2),$$

and their angular relations to the internal bisector of the angle A by θ and θ_1.

If now these lines be conceived as drawn parallel respectively to the given lines,

$$l\alpha + m\beta + n\gamma = 0,$$
$$l_1\alpha + m_1\beta + n_1\gamma = 0,$$

whose condition of perpendicularity is sought, we may write, regarding only the ratios of β and γ,

$$(ma - lb)\beta + (na - lc)\gamma = 0,$$

$$(m_1a - l_1b)\beta + (n_1a - l_1c)\gamma = 0,$$

which are of the form of (1) and (2).

Equation (1) may be treated as follows:—

$$\sin\left(\frac{A}{2}+\theta\right) : \sin\left(\frac{A}{2}-\theta\right) :: t : s.$$

This, by composition, division, alternation, and reducing, becomes

$$\tan\theta : \tan\frac{A}{2} :: t-s : t+s.$$

Similarly, $$\tan\theta_1 : \tan\frac{A}{2} :: t_1-s_1 : t_1+s_1.$$

But the condition of perpendicularity in general is

$$\tan\theta\tan\theta_1+1 = 0;$$

therefore, by reduction and supplying values, we get

$$\begin{aligned} mm_1a^2+nn_1a^2+ll_1(b^2+c^2-2bc\cos A) \\ -(nl_1+n_1l)(ac-ab\cos A)-(lm_1+l_1m)(ab-ac\cos A) \\ -(mn_1+m_1n)(a^2\cos A) = 0, \end{aligned}$$

which, remembering that

$$b^2+c^2-2bc\cos A = a^2, \qquad c-b\cos A = a\cos B,$$
$$b-c\cos A = a\cos C,$$

becomes

$$\begin{aligned} ll_1-(mn_1+m_1n)\cos A + mm_1-(nl_1+n_1l)\cos B \\ + nn_1-(lm_1+l_1m)\cos C = 0, \end{aligned}$$

the condition necessary.

General Exercises.

1. To prove whether perpendiculars upon the opposite sides meet.

We perceive that the perpendicular divides any angle of the triangle into parts which are the complements of the remaining two angles.

Therefore the equation of AD is

$$\cos B\,.\,\beta = \cos C\,.\,\gamma,$$

or, more fully,

$$\beta : \gamma :: \sin CAD : \sin BAD :: \cos C : \cos B.$$

Similarly, $$\cos A\,.\,\alpha = \cos B\,.\,\beta,$$

and $$\cos C\,.\,\gamma = \cos A\,.\,\alpha.$$

If we write the equations of these perpendiculars in order, we see that α does not appear in the first or AD, β in the second or BE, and γ is wanting in the last or CF; and remembering that these are the coefficients of a linear equation,

as, $$0\alpha + \cos B\,.\,\beta - \cos C\,.\,\gamma = 0,$$
$$\&c. \qquad \&c.,$$

and remembering also that, by Art. 8, the problem is simply elimination between these three equations, the condition of concurrence, as we have already seen, is presented by the determinant

$$\begin{vmatrix} 0 & \cos B & -\cos C \\ -\cos A & 0 & \cos C \\ \cos A & -\cos B & 0 \end{vmatrix} = 0.$$

2. On the sides of the triangle of reference, as bases, are constructed three triangles, similar and so placed that the adjacent base angles are equal, and each base angle respectively equal to the vertex most remote; thus :

$$A_1BC = AB_1C = ABC_1, \qquad B_1CA = BC_1A = BCA_1,$$

and $$C_1AB = CA_1B = CAB_1;$$

then will AA_1, BB_1, CC_1 cointersect.

Since the point A_1 falls without the triangle of reference, but within the angle A, the ordinate α must be negative. The same applies to β at the point B_1, &c.

We first seek the perpendiculars on a, b, c from A_1, which are, in order,

$$S\,.\,\sin C_1, \qquad S\,.\,\sin (C+C_1), \qquad S_1\,.\,\sin (B+B_1),$$

where $$S = \frac{a \sin B_1}{\sin A_1}, \text{ and } S_1 = \frac{a \sin C_1}{\sin A_1}.$$

Dividing these by the first to obtain the ratios, we have for the coordinates

of A_1,	$-1, \quad h, \quad g,$
of B_1,	$h, \quad -1, \quad f,$
of C_1,	$g, \quad f, \quad -1,$

where f represents $\dfrac{\sin(A+A_1)}{\sin A_1}$,

" g " $\dfrac{\sin(B+B_1)}{\sin B_1}$,

" h " $\dfrac{\sin(C+C_1)}{\sin C_1}$.

The ratios of A are	1, 0, 0,
" B "	0, 1, 0,
" C "	0, 0, 1.

Hence the equation of the line joining the two points A and A_1 is (Art. 7)

$$0 . \alpha + g . \beta - h . \gamma = 0;$$

for BB_1, $$-f . \alpha + 0 . \beta + h . \gamma = 0;$$

for CC_1, $$f . \alpha - g . \beta + 0 . \gamma = 0.$$

By (Art. 9) the determinant of concurrence is formed from these three lines; that is,

$$\begin{vmatrix} 0 & g & -h \\ -f & 0 & h \\ f & -g & 0 \end{vmatrix} = 0.$$

3. In the same manner, from the same figure, prove that (BC, B_1C_1), (CA, C_1A_1), (AB, A_1B_1) respectively meet in points which are collinear.

15. *A straight line through a given point and parallel to a given straight line.*

Let (l, m, n) be the given straight line, (f, g, h) the given point, (l_1, m_1, n_1) the required straight line.

The condition of parallelism of

$$l_1\alpha + m_1\beta + n_1\gamma = 0 \quad \ldots\ldots\ldots\ldots (1),$$

and

$$l\,\alpha + m\,\beta + n\,\gamma = 0 \quad \ldots\ldots\ldots\ldots (2),$$

by (Art. 10), is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l & m & n \\ a & b & c \end{vmatrix} = 0;$$

that is,

$$l_1\begin{vmatrix} m & n \\ b & c \end{vmatrix} + m_1\begin{vmatrix} n & l \\ c & a \end{vmatrix} + n_1\begin{vmatrix} l & m \\ a & b \end{vmatrix} = 0 \ldots\ldots\ldots (3).$$

If the locus passes through f, g, h, we must have

$$l_1 f + m_1 g + n_1 h = 0 \quad \ldots\ldots\ldots\ldots (4).$$

We are now furnished with three equations to eliminate $l_1 m_1 n_1$; viz., (1), (3), and (4).

Hence

$$\begin{vmatrix} \alpha & \beta & \gamma \\ f & g & h \\ A & B & C \end{vmatrix} = 0$$

is the equation sought, where A, B, C stand for the minors of (3).

16. To show that $\begin{vmatrix} b & c \\ R & S \end{vmatrix} = 2\Delta(\alpha - \alpha_1)$,

where

$$R = \begin{vmatrix} \gamma & \alpha \\ \gamma_1 & \alpha_1 \end{vmatrix}, \quad S = \begin{vmatrix} \alpha & \beta \\ \alpha_1 & \beta_1 \end{vmatrix}$$

are the second and first determinants formed from the coordinates of two points (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$,

$$\begin{vmatrix} b & c \\ R & S \end{vmatrix} = \begin{vmatrix} 0 & -c & b \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -1 & 0 \\ \alpha & \beta & 2\Delta \\ \alpha_1 & \beta_1 & 2\Delta \end{vmatrix} = 2\Delta(\alpha - \alpha_1).$$

Similarly, if $$Q = \begin{vmatrix} \beta & \gamma \\ \beta_1 & \gamma_1 \end{vmatrix},$$

$$\begin{vmatrix} c & a \\ S & Q \end{vmatrix} = 2\Delta(\beta-\beta_1), \quad \text{and} \quad \begin{vmatrix} a & b \\ Q & R \end{vmatrix} = 2\Delta(\gamma-\gamma_1).$$

17. *Deduced coordinates of the triangle of reference.*

1st. *Of the angular points.*

At A, $$\beta = 0, \quad \gamma = 0.$$

Hence $$a\alpha = 2\Delta, \quad \alpha = \frac{2\Delta}{a}.$$

At B, similarly, $$0, \quad \frac{2\Delta}{b}, \quad 0.$$

At C, $$0, \quad 0, \quad \frac{2\Delta}{c}.$$

2nd. *Of the middle point of BC.*

Evidently $$b\beta = \text{area of triangle} = c\gamma,$$
and $$\alpha = 0,$$
Hence $0, \ \frac{\Delta}{b}, \ \frac{\Delta}{c}$ are the coordinates.

3rd. *Of the foot of the perpendicular from A upon BC.*

The perpendicular $= \frac{2\Delta}{a}$, by 1st case.

Hence $$\frac{2\Delta}{a}\cos C = \beta, \quad \frac{2\Delta}{a}\cos B = \gamma;$$
and $$0, \quad \frac{2\Delta}{a}\cos C, \quad \frac{2\Delta}{a}\cos B$$
are the required coordinates.

4th. *Of the centre of the inscribed circle.*

The point being equally distant from the three lines of reference, we must have

$$\alpha = \beta = \gamma = \frac{2\Delta}{a+b+c}.$$

Ex.—Prove that

$$\alpha = r\cos A, \quad \beta = r\cos B, \quad \gamma = r\cos C,$$

r being radius of circumscribed circle.

18. *Distance between two points.*

Various expressions may be deduced. One only is here given; others will be given hereafter.

Let B_1C_1, B_1A_1, drawn parallel to the sides of the triangle of reference BC, BA respectively, be two sides of a quadrilateral $B_1C_1PA_1$ inscribed in a circle whose diameter is B_1P; (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ the coordinates of B_1, P; $r =$ the distance between them. Through C_1 draw a diameter C_1D. Join A_1C_1, DA_1.

$$\begin{aligned} A_1C_1^2 &= PC_1^2 + PA_1^2 - 2PC_1 . PA_1 \cos C_1PA_1 \\ &= PC_1^2 + PA_1^2 + 2PC_1 . PA_1 \cos B \;\ldots\ldots\ldots\ldots (1). \end{aligned}$$

The angle at $D =$ the angle $C_1B_1A_1 = B$,

$$A_1C_1 = C_1D \sin B = PB_1 \sin B = r\sin B,$$

$$PC_1 = \alpha - \alpha_1, \quad PA_1 = \gamma - \gamma_1.$$

Substituting these values in (1),

$$r^2 \sin^2 B = (\alpha-\alpha_1)^2 + (\gamma-\gamma_1)^2 + 2(\alpha-\alpha_1)(\gamma-\gamma_1)\cos B,$$

which is the required equation.

This, however, may be made symmetrical with the determinants formed from the coordinates of the points B_1, P. These determinants are represented in (Art. 16) by Q, R, S; also in the same Article it was shown that

$$\frac{\begin{vmatrix} b & c \\ R & S \end{vmatrix}}{2\Delta} = \alpha - \alpha_1, \quad \text{and} \quad \frac{\begin{vmatrix} a & b \\ Q & R \end{vmatrix}}{2\Delta} = \gamma - \gamma_1.$$

Hence $$4\Delta^2 r^2 \sin^2 B = X^2 + Y^2 + 2XY\cos B,$$

where $$X = \begin{vmatrix} b & c \\ R & S \end{vmatrix}, \quad Y = \begin{vmatrix} a & b \\ Q & R \end{vmatrix}.$$

Developing the values of X and Y, we find

$$4\Delta^2 r^2 \sin^2 B = b^2(Q^2+R^2+S^2-2RS\cos A-2SQ\cos B -2QR\cos C).$$

By (Art. 5), $\qquad 2\Delta \sin B = Vb.$

Therefore

$$r = \frac{1}{V}\sqrt{(Q^2+R^2+S^2-2RS\cos A-2SQ\cos B-2QR\cos C)}.$$

19. *The area of a triangle from the trilinear coordinates of the angular points.*

The area of a triangle expressed as a determinant in Cartesian coordinates, the axes being rectangular, is (Art. 10, D.)

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

which, referred to oblique axes, becomes

$$\frac{1}{2}\operatorname{cosec}\omega \begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix}$$

Let $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ take the places of x and y, and multiply and divide by V, and we shall have

$$\frac{\operatorname{cosec} C}{2V}\begin{vmatrix} V & V & V \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

Multiply the last row by $\sin A$ and the second by $\sin B$, and then take the sum of these new rows from the first.

Observing (Art. 5) that

$$\alpha \sin A + \beta \sin B + \gamma \sin C = V,$$

we are enabled to write

$$\text{Area} = \frac{\operatorname{cosec} C}{2V}\begin{vmatrix} \gamma_1 \sin C & \gamma_2 \sin C & \gamma_3 \sin C \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

$$= \frac{1}{2V}\begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix}$$

or, more symmetrically,

$$= \frac{1}{2V}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

It will be observed that the angle ω, between the axes, is changed to one of the angles of the triangle of reference in passing from the Cartesian to the trilinear system.

20. *Perpendicular distance of a point from a line.*

Let the coordinates of the point be $(\alpha_2, \beta_2, \gamma_2)$; (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ the coordinates of two points in the line; r the distance between these last-named points; and p the perpendicular sought.

Then pr = twice the area of the triangle of which the three points are the vertices.

By the preceding Article,

$$p = \frac{1}{r}\cdot\frac{1}{V}D.$$

But, by (Art. 18), $\quad r = \frac{1}{V}Z,$

where Z = the radical part in the final value of r.

Hence $\quad p = \frac{D}{Z} = \frac{\begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}}{Z}$

$$= \frac{Q\alpha_2 + R\beta_2 + S\gamma_2}{Z} \quad \text{..........................(1).}$$

We have already seen (Art. 7) that the equation to a line joining two points is

$$l\alpha+m\beta+n\gamma=0.$$

But the numerator of (1) is, in fact, the same expression under another form. As general equations of a straight line joining two points they must be identical.

Hence we may write

$$p=\frac{l\alpha+m\beta+n\gamma}{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)^{\frac{1}{2}}}.$$

CHAPTER II.

THE EQUATION IN TERMS OF PERPENDICULARS—TANGENTIAL AND TRIANGULAR COORDINATES—IMAGINARIES.

21. It is necessary, as we proceed, to introduce the equation of the straight line under somewhat different forms. We have considered a point as determined by its perpendicular distances from the three sides of the triangle of reference. A line joining two of these points has thus far occupied our attention.

Let now the perpendicular distances of the three angular points A, B, C from a straight line be p, q, r, and let it be required to find the equation to this straight line in terms of these quantities.

We will assume two points on this straight line, one upon each side of the perpendicular p; $d =$ the distance between them.

$\frac{pd}{2} =$ area of the triangle formed by these two points and the point A.

For the coordinates of the point A we have (Art. 17)

$$\frac{2\Delta}{a}, \quad 0, \quad 0.$$

Let (α, β, γ), $(\alpha_1, \beta_1, \gamma_1)$ be the coordinates of the two given points.

Hence (Art. 19)

$$pd = \frac{1}{V}\begin{vmatrix} \frac{2\Delta}{a} & 0 & 0 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

also $$qd = \frac{1}{V}\begin{vmatrix} 0 & \frac{2\Delta}{b} & 0 \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

and $$rd = \frac{1}{V}\begin{vmatrix} 0 & 0 & \frac{2\Delta}{c} \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}$$

Multiplying these equations by $a\alpha$, $b\beta$, $c\gamma$ respectively, and adding, we have (D. 7)

$$(ap\alpha + bq\beta + cr\gamma)\, d = \frac{2\Delta}{V}\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} = 0,$$

that is, $$ap\alpha + bq\beta + cr\gamma = 0.$$

This is only another form, or a special case, of

$$l\alpha + m\beta + n\gamma = 0,$$

in which l, m, n are proportional to ap, bq, cr, or to the determinants of (Art. 7, eq. 1).

22. *The perpendicular distance of a point from the line*

$$ap\alpha + bq\beta + cr\gamma = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Let the given point be (f, g, h), through which a parallel is drawn; d the perpendicular distance required.

Then the distances from A, B, C to this parallel will be represented by the perpendiculars $d \pm p$, $d \pm q$, $d \pm r$, which, substituted in (1), give

$$a\,(d \pm p)\,\alpha + b\,(d \pm q)\,\beta + c\,(d \pm r)\,\gamma = 0.$$

But, by hypothesis, this line passes through (f, g, h).

Hence $$a\,(d \pm p) f + b\,(d \pm q)\, g + c\,(d \pm r)\, h = 0,$$

$$(af + bg + ch)\, d = \mp (apf + bqg + crh),$$

which gives $$d = \pm \frac{apf + bqg + crh}{2\Delta},$$

an equation for the perpendicular distance.

23. The equation
$$d = \frac{apf + bqg + crh}{2\Delta}$$

evidently gives the altitude of a triangle whose vertex is f, g, h, and the equation of the base
$$apf + bqg + crh = 0.$$

The equations of the sides will differ only in the perpendicular; hence these may be written
$$ap_1 f + bq_1 g + cr_1 h = 0,$$
$$ap_2 f + bq_2 g + cr_2 h = 0.$$

With these two equations, and
$$af + bg + ch = 2\Delta,$$

the values of f, g, h may be determined; that is,
$$af : bg : ch : 2\Delta :: \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix} : \begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}$$

$$\frac{af}{2\Delta} = \frac{\begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

Multiplying this equality by p, and the second and third equations formed from the above proportion by q and r respectively, and adding, we have
$$d = \frac{apf + bqg + crh}{2\Delta} = \frac{\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}.$$

24. We will now show the method of expressing the position of a right line by coordinates, and that of a point by an equation.*

Let p, q, r be the unknown, α, β, γ the known, coordinates; then, by the equation we have just considered, we are enabled to determine a relation between p, q, r which will be true for any right line drawn through the fixed point of which α, β, γ are the coordinates; that is,

$$a\alpha p + b\beta q + c\gamma r = 0,$$

which is called an equation, in tangential coordinates, of the point whose trilinear coordinates are α, β, γ.

25. *To find the tangential equation to the point of intersection of two right lines.*

Let (p_1, q_1, r_1), (p_2, q_2, r_2) be the tangential coordinates of the two lines; (α, β, γ) the trilinear coordinates of their point of intersection. Evidently, then, (α, β, γ) is a point on each of two lines whose perpendicular distances from A, B, C are p_1, q_1, r_1; p_2, q_2, r_2.

We first determine the ratios of the trilinear coordinates of the point.

We have (Art. 21)

$$ap_1\alpha + bq_1\beta + cr_1\gamma = 0,$$

and

$$ap_2\alpha + bq_2\beta + cr_2\gamma = 0;$$

hence
$$a\alpha : b\beta : c\gamma :: \begin{vmatrix} q_1 & r_1 \\ q_2 & r_2 \end{vmatrix} : \begin{vmatrix} r_1 & p_1 \\ r_2 & p_2 \end{vmatrix} : \begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

Multiply each of these ratios by p, q, r respectively, and add; then each of these ratios

$$= \frac{a\alpha p + b\beta q + c\gamma r}{\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix}}$$

* Salmon's Conics, p. 65.

The numerator expresses a relation between p, q, r by the preceding Article; but the denominator evidently expresses the same relation.

Hence
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0$$
is the equation required.

Otherwise, suppose
$$a\alpha p + b\beta q + c\gamma r = 0 \quad \text{.....................} \quad (1)$$
the equation of the point of intersection, which must be satisfied by the coordinates of any line drawn through that point; but (p_1, q_1, r_1), (p_2, q_2, r_2) by hypothesis are perpendiculars from the points of reference upon lines drawn through the point of intersection.

Hence
$$a\alpha p_1 + b\beta q_1 + c\gamma r_1 = 0 \quad \text{.....................} \quad (2),$$
$$a\alpha p_2 + b\beta q_2 + c\gamma r_2 = 0 \quad \text{.....................} \quad (3);$$
that is, (1), (2), (3) furnish the determinant
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \end{vmatrix} = 0,$$
the same relation and equation as before.

26. *Tangential equation of a point at infinity.*

The point is clearly the intersection of two parallels.

Let (p_1, q_1, r_1) and (p_1+t, q_1+t, r_1+t) be the parallels. But the condition of parallelism (Art. 10) is
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ p_1+t & q_1+t & r_1+t \end{vmatrix} = 0,$$
which may be written
$$\begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ t & t & t \end{vmatrix} = \begin{vmatrix} p & q & r \\ p_1 & q_1 & r_1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

The last determinant identically vanishes, as will be seen, if a common factor can be taken so as to make the first row unity; in other words, if $p=q=r$. That is, points at infinity are comprised upon the line

$$p=q=r;$$

and equation (1) of last Article reduces to

$$a\alpha+b\beta+c\gamma=0,$$

a relation which has already been interpreted (Art. 11).

27. Since we define the equation to a point in these coordinates as an equation satisfied by the coordinates of all right lines drawn through the point, it follows that, if

$$L=0,$$
$$V=0,$$

be two equations representing two points in tangential coordinates, then the equation

$$L+kV=0$$

being satisfied, as it evidently is, by the coordinates of L and V, must express a point on the line joining the given points.

28. Reserving for the present the further development and the application of tangential coordinates, we will just mention a system of coordinates known by the term *triangular*.

Instead of the trilinear equation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

we may write $$\frac{a\alpha}{2\Delta}+\frac{b\beta}{2\Delta}+\frac{c\gamma}{2\Delta}=1;$$

and, denoting the ratios of the left member by x, y, z, we have

$$x+y+z=1.$$

The ratios $\frac{a\alpha}{2\Delta}$, $\frac{b\beta}{2\Delta}$, $\frac{c\gamma}{2\Delta}$ evidently represent the ratios of the triangles BPC, APC, APB to the triangle of reference.

It is clear how the trilinear coordinates α, β, γ are related to x, y, z; for, if we divide x by $a\alpha$, we have $\dfrac{1}{2\Delta}$. In the same manner, y divided by $b\beta = \dfrac{1}{2\Delta}$; so that

$$\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma} = \frac{1}{2\Delta}.$$

29. The coordinates of the middle point of BC are, in triangular coordinates, $\qquad 0, \ \tfrac{1}{2}, \ \tfrac{1}{2}.$

This appears, since $\qquad b\beta = \Delta.$

But $\qquad b\beta = 2\Delta y;$

hence $\qquad y = \tfrac{1}{2};$

similarly, $\qquad z = \tfrac{1}{2};$

while $\qquad x = 0.$

30. We have seen (Art. 17) that the coordinates of the foot of the perpendicular from A upon BC are

$$0, \quad \frac{2\Delta}{a}\cos C, \quad \frac{2\Delta}{a}\cos B.$$

These expressions, transformed as above, become

$$0, \quad \frac{b\cos C}{a}, \quad \frac{c\cos B}{a}.$$

Referring again to (Art. 17), we find the coordinates of the centre of the inscribed circle, which, transferred into the triangular system, become

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{1}{a+b+c}.$$

Transforming the area of a triangle (Art. 19), we have

$$\frac{1}{2V}\cdot\frac{8\Delta^3}{abc}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

= (Art. 5)

$$\frac{8\Delta^3}{abc}\cdot\frac{1}{2(\alpha\sin A+\beta\sin B+\gamma\sin C)}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \Delta\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

The equation to a straight line joining two points is, in a similar manner, found to be (Art. 7)

$$\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$

The condition of concurrence is the same for both systems.

The equation to the right line at infinity (Art. 11) becomes in triangular coordinates

$$x+y+z=0;$$

and consequently the condition of parallelism (Art. 10) is easily transformed to

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

The equation to the perpendicular (Art. 14, Ex. 1)

$$\beta\cos B=\gamma\cos C,$$

transformed from trilinear to triangular coordinates, is

$$y\cot B=z\cot C.$$

Thus a great number of similar transformations might be written out.

These, however, must suffice for the present, and these are probably sufficient to give the reader a correct idea of such changes when they become necessary. The useful applications of these several systems of coordinates must be learned chiefly from a study of lines of a higher order than the first.

31. The principle detailed by Dr. Salmon* is equally ap-

* Conics, p. 33.

plicable to trilinear or triangular coordinates, or any system in which a point is determined by coordinates; that is, if

$$u = l\alpha + m\beta + n\gamma = 0,$$

and $$v = l_1\alpha + m_1\beta + n_1\gamma = 0,$$

then will $$u + kv = 0 \quad \text{..........................(1)}$$

represent a line passing through the intersection of u and v, which line, it is evident, can be made to represent any particular line by giving particular values to the arbitrary constant k.

Let us try a simple application.

Suppose the triangle of reference circumscribed by lines whose equations are

$$u = 0, \quad v = 0, \quad w = 0;$$

that is, representing A_1B_1, B_1C_1, C_1A_1. Let A_1B_1 be produced to some point B_2, and from B_2 let B_2C_2 be drawn, and let B_2C_2 be the line whose equation is to be determined. Join B_2C_1. But C_1 is the point of intersection of $v=0$ and $w=0$. Hence, from what has just preceded, B_2C_1 will be represented by

$$v + kw = 0.$$

Also, since B_2C_1 and A_1B_1 (produced) meet in B_2, the line B_2C_2, which is drawn through their intersection, will, by the same considerations, be represented by

$$k_1u + v + kw = 0,$$

which, written symmetrically and in the usual form, becomes

$$\lambda u + \mu v + \nu w = 0.$$

It is manifest that this proof is not restricted to lines forming a triangle. It is equally plain that they should not be parallel.

32. In order that a point may be determined upon the line

$$l\alpha + m\beta + n\gamma = 0,$$

its coordinates must simultaneously satisfy the relation

$$a\alpha + b\beta + c\gamma = 2\Delta.$$

If such values prove to be irrational, they are, by convention, said to be the coordinates of an imaginary point. Since quadratics involve two roots—sometimes imaginary—there will be the same number of intersections, if the question is one of intersection, real or imaginary,—or, more exactly, real, coincident, or imaginary. But as this truth is so well known and so fully exhibited in Cartesian Geometry, we shall here consider only what is peculiar to our subject.

33. It is evident that the imaginary roots of a trilinear equation of the second degree must be of the form

$$\alpha+\alpha_1\sqrt{-1}, \quad \beta+\beta_1\sqrt{-1}, \quad \gamma+\gamma_1\sqrt{-1} \ \ldots\ldots\ldots(a).$$

Suppose these roots to be the coordinates of an imaginary point. Then, by the last Article, these must satisfy the relation

$$a\alpha+b\beta+c\gamma = 2\Delta.$$

Making the substitution, we have

$$(a\alpha+b\beta+c\gamma)+(a\alpha_1+b\beta_1+c\gamma_1)\sqrt{-1} = 2\Delta \ \ldots\ldots(1);$$

wherefore $\qquad a\alpha_1+b\beta_1+c\gamma_1 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$

and $\qquad a\alpha+b\beta+c\gamma = 2\Delta \ \ldots\ldots\ldots\ldots\ldots\ldots(3).$

From which we see that (1) is made up of both real and imaginary parts; the imaginary parts satisfying (2) the equation to the line at infinity (Art. 11); while (3) is of course satisfied by its own coordinates. The reader will learn to distinguish between the coordinates of an imaginary point and those of an imaginary point at infinity; that is, if the coordinates (a) had been regarded as the coordinates of an imaginary point (or proportional to them) at infinity, both (2) and (3) must have been written $=0$.

34. The equation to an imaginary right line may be written

$$(l+l_1\sqrt{-1})f+(m+m_1\sqrt{-1})g+(n+n_1\sqrt{-1})h = 0.$$

35. Writers upon equations of the second degree representing right lines in Cartesian coordinates are accustomed to dispose of the contingency of two imaginary roots by referring both to two imaginary lines drawn through the origin, thus determining a *real* point. So now we say that every imaginary right line passes through one real point, and but one.

If we consider the equation of (Art. 34), we see that the real and imaginary parts are not coincident, and consequently the factor $\sqrt{-1}$ does not divide out; hence the equation may be expressed

$$u+v\sqrt{-1}=0 \text{ ....(1)},$$

in which u and v are functions of the coordinates of the given straight line. This equation is manifestly entirely similar to equation (1), (Art. 31).

It is also to be observed that u and v are of the first degree, and hence

$$u=0$$

and

$$v=0$$

are satisfied by real values, which values satisfy (1), which passes through the point of intersection of u and v, and therefore each straight line passes through a *real point*.

36. Suppose the equation to a straight line

$$lf+mg+nh=0$$

to pass through an imaginary point whose coordinates are given in [Art. 33 (a)]; then

$$l\alpha+m\beta+n\gamma+(l\alpha_1+m\beta_1+n\gamma_1)\sqrt{-1}=0;$$

and consequently

$$l\alpha+m\beta+n\gamma=0,$$

$$l\alpha_1+m\beta_1+n\gamma_1=0,$$

which equations determine the ratios of l, m, n; or we may determine them fully by the determinant

$$\begin{vmatrix} f & g & h \\ \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix}=0,$$

which is the equation to the straight line drawn through the imaginary point whose coordinates are

$$\alpha+\alpha_1\sqrt{-1},\quad \beta+\beta_1\sqrt{-1},\quad \gamma+\gamma_1\sqrt{-1};$$

and consequently

$$\alpha-\alpha_1\sqrt{-1},\quad \beta-\beta_1\sqrt{-1},\quad \gamma-\gamma_1\sqrt{-1}.$$

Therefore, since imaginary roots enter by pairs into an equation, the imaginary points of intersection of two lines (curves) *will be found upon real straight lines by twos.*

37. If we have an equation of the form

$$l\beta^2-m\beta\gamma+n\gamma^2=0 \quad \text{..................(1)},$$

we can evidently subject it to the same reasoning which is applied to the quadratic

$$x^2-pxy+qy^2=0.^*$$

Each equation is reducible to the form

$$(\beta-s\gamma)(\beta-s_1\gamma)=0;$$

that is, the two straight lines

$$\beta-s\gamma=0 \quad \text{..........................(2)},$$

$$\beta-s_1\gamma=0 \quad \text{..........................(3)},$$

are real or imaginary according as we find, by the resolution of (1) for the ratio $\beta:\gamma$, that $4ln$ is less or greater than m^2.†

Examining (2) and (3) in the light of (Art. 31), we see that these lines intersect in the point A of the triangle of refernece.

38. *Excursus on imaginary right lines and points.*

It is evident, from what has immediately preceded, that this portion of the subject is capable of considerable expansion, and that this system of coordinates is eminently fitted to deal with the Infinite and Imaginary. From what has already been said in reference to the adaptation of the reasoning

* Salmon's Conics, p. 69.

† Algebra, Bourdon, p. 159.

employed in Cartesian methods to trilinear coordinates, the views of high authorities upon these results are interesting.

Poncelet* has discovered and illustrated geometrically the *rationale* of the principles which, upon purely analytical grounds, we are enabled to re-discover, apply, and extend; he has pointed out the correspondence of points, some real and some imaginary, and taught that theorems concerning imaginary points and lines may be extended to real points and lines, and hence shown how to indicate the properties of a figure when some of the lines and points are real and some imaginary. By the method of trilinear coordinates we are enabled quickly to generalize all those theorems which are concerned with the line at infinity. For example, if four points on a conic, or four tangents to a conic, are given, and it is required to find the locus of the centre of the conic, we proceed to find the locus of the pole of the line

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

which also gives us, the conditions being the same, the locus of the pole of any line

$$\lambda\alpha + \mu\beta + \nu\gamma = 0.$$

In applying the method of projections, the analytic shows its superiority over the synthetic method, by proving the general theorem at once, rather than by inferring it by the projection from a more elementary state of the figure.

As to the results reached in our discussion of parallelism, and what we have said upon the theory and use of the line

$$\alpha \sin A + \beta \sin B + \gamma \sin C = 0,$$

nothing is affirmed beyond what has been received, almost without dissent, from the first, both upon geometrical and analytical considerations. See Chasles (Géom. Sup.), Townsend (Vol. I., p. 16, Art. 136), Salmon (Conics, pp. 64, 318), Poncelet (Proj. Persp., p. 53), Hamilton (Quaternions, p. 90).

* Traité des Propriétés Projectives des Figures.

38. *Tangent of angle between two lines.*

Let
$$l\alpha+m\beta+n\gamma=0,$$
$$l_1\alpha+m_1\beta+n_1\gamma=0,$$
be the given lines.

If θ, θ_1 be their respective inclinations to one of the lines of reference, then, by the reasoning in Art. 14, we must have as the tangent of the difference of the two angles, that is, the tangent of the required angle,

$$\tan(\theta-\theta_1)=\frac{\tan\theta-\tan\theta_1}{1+\tan\theta\tan\theta_1},$$

which becomes, by a laborious reduction,

$$\frac{\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ \sin A & \sin B & \sin C \end{vmatrix}}{\mathrm{P}}$$

where P is the sinister member of the equation of perpendicularity given on page 17.

This is probably the simplest form possible in trilinear coordinates.

Examples under Chapters I. and II.

1. Express the parallelism of

$$l\alpha+m\beta+n\gamma=0,$$

with AC in the triangle of reference.

Ans.:
$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & 1 & 0 \end{vmatrix}=0.$$

2. The same line with BC; with AB.

3. What relation of two lines is expressed by the determinant

$$\begin{vmatrix} a & b & c \\ 1 & 0 & \cos B \\ 0 & 1 & \cos A \end{vmatrix} = 0;$$

and what are the lines?

4. What condition is expressed by

$$\begin{vmatrix} a & b & c \\ l & m & n \\ 0 & -1 & 1 \end{vmatrix} = 0?$$

5. Find the angle between the lines

$$\alpha = \gamma \cos B,$$

and

$$\beta = \gamma \cos A.$$

6. If $u+v\sqrt{-1}=0$, $u'+v'\sqrt{-1}=0$ are imaginary straight lines having a real point of intersection, then the four real straight lines $u=0$, $v=0$, $u'=0$, $v'=0$ are concurrent.

7. What is the determinant expressing the equation of the right line drawn through the intersections of the pairs of lines

$$2au+bv+cw=0, \quad bv+cw=0;$$

$$2bu+av+cw=0, \quad av-cw=0?$$

CHAPTER III.

THE TRILINEAR METHOD APPLIED TO CONICS.

39. We will now call attention to the fact, which may not have escaped the notice of the reader, that trilinear equations are always homogeneous. If not so in form, they can be made so by a very simple process. Since

$$a\alpha+b\beta+c\gamma = 2\Delta,$$

we may write
$$\frac{a\alpha+b\beta+c\gamma}{2\Delta} = 1;$$

and therefore any term of an equation may be multiplied by this fraction without affecting the pre-existing relation of equality. Thus, if we have

$$\alpha^2-2\alpha\beta+\gamma = 2,$$

we may proceed to raise each non-homogeneous term to the second order, as

$$\alpha^2-2\alpha\beta+\gamma\left(\frac{a\alpha+b\beta+c\gamma}{2\Delta}\right) = 2\left[\frac{a\alpha+b\beta+c\gamma}{2\Delta}\right]^2.$$

40. Another consideration, which has been referred to, may be here emphasized; viz., that we are not concerned with the absolute values of the coordinates, but with their ratios; and this advantage we derive from the principle of homogeneity which belongs to every trilinear equation; thus,

$$\alpha^2-2\alpha\beta+\gamma^2 = 0$$

is, in fact,
$$\left(\frac{\alpha}{\gamma}\right)^2-2\left(\frac{\alpha}{\gamma}\right)\cdot\frac{\beta}{\gamma}+1 = 0,$$

in which only the ratios $\frac{\alpha}{\gamma}$ and $\frac{\beta}{\gamma}$ appear. Beyond these ratios it is not necessary for us to inquire.

41. It may be desirable to find the equation to the same locus, but referred to another triangle of reference.

First Transformation,

when the equations of the sides of the new triangle are given.

These sides being represented by equations in terms of the perpendiculars from the angular points of the original triangle, we have (Art. 21)

$$\begin{array}{lll} \text{coordinates of } A, & (p, p_1, p_2)\,; \\ \quad\quad ,, \quad\quad B, & (q, q_1, q_2)\,; \\ \quad\quad ,, \quad\quad C, & (r, r_1, r_2)\,; \end{array}$$

that is,

$$apf + bqg + crh = 0 \quad \ldots\ldots\ldots\ldots (1),$$
$$ap_1f + bq_1g + cr_1h = 0 \quad \ldots\ldots\ldots\ldots (2),$$
$$ap_2f + bq_2g + cr_2h = 0 \quad \ldots\ldots\ldots\ldots (3),$$

where (f, g, h) are the old coordinates of any point P.

To find the locus of the homogeneous equation

$$F(f, g, h) = 0.$$

When referred to (1), (2), (3), we observe that f represents the perpendicular from (f_1, g_1, h_1)—these being the new coordinates of P—on the line joining B and C. Therefore

$$(f_1, g_1, h_1), \quad (q, q_1, q_2), \quad (r, r_1, r_2)$$

indicate the angular points of a triangle whose area is found by Art. 19,

$$\text{double area} = af = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ q & q_1 & q_2 \\ r & r_1 & r_2 \end{vmatrix}.$$

Similarly, $bg = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ r & r_1 & r_2 \\ p & p_1 & p_2 \end{vmatrix}$, $ch = \frac{1}{V_1}\begin{vmatrix} f_1 & g_1 & h_1 \\ p & p_1 & p_2 \\ q & q_1 & q_2 \end{vmatrix}$, from which the values of f, g, h are readily determined. Hence, representing these determinants by Q, R, S respectively, we may write

$$F\left(\frac{Q}{a}, \frac{R}{b}, \frac{S}{c}\right) = 0$$

as the equation with new lines of reference, the degree not being changed by transformation.

42. *Second Transformation,*

coordinates of the new points of reference being given.

A triangle drawn within or without the original triangle will sufficiently represent the construction.

Let the perpendiculars from A_1, B_1, C_1, the new points of reference, upon BC be denoted by p, p_1, p_2; on AC by q, q_1, q_2; on AB by r, r_1, r_2; a_1, b_1, c_1 the sides of the new triangle; f_1, g_1, h_1 the new coordinates, and f, g, h the old coordinates, of any point P. Then, by Art. 21, we find

$$a_1 p f_1 + b_1 p_1 g_1 + c_1 p_2 h_1 = 0,$$
$$a_1 q f_1 + b_1 q_1 g_1 + c_1 q_2 h_1 = 0,$$
$$a_1 r f_1 + b_1 r_1 g_1 + c_1 r_2 h_1 = 0.$$

Representing these equations by Q, R, S respectively, we have, by Art. 22, the distance of P from each of the sides of the original triangle expressed in a simple form; that is,

$$f = \frac{Q}{2\Delta_1}, \quad g = \frac{R}{2\Delta_1}, \quad h = \frac{S}{2\Delta_1},$$

the old coordinates expressed in terms of the new.

43. We shall now pass on to the consideration of curves of the second degree. An important property of these curves was conceived by the early geometers; viz., that every curve of this degree might be regarded as a conic section. What then can be easily shown may be stated here, that the section of a right circular cone by any plane can be expressed by a

homogeneous equation of the second degree in trilinear coordinates. This can be readily proved by selecting particular lines of reference; and since, by the preceding Articles, we may transform to any other lines without affecting the degree of the equation, we may regard this as a general truth irrespective of the lines of reference.

Let us write

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0$$

as the general equation of the second degree in trilinear coordinates.

This equation, it will be seen, contains six terms; but as the nature of the curve does not depend upon the independent magnitude of these coefficients, we may simply regard their mutual ratios, or, in other words, assign a particular value to one of the coefficients, varying the values of the others.

Here, then, as in the Cartesian coordinates, we can find the equation to the conic described through five points. There are, in other words, five constants to be determined whose values substituted in the general equation will give the equation of the conic through five points; that is,

$$\begin{vmatrix} \alpha^2 & \beta^2 & \gamma^2 & \beta\gamma & \gamma\alpha & \alpha\beta \\ \alpha_1^2 & \beta_1^2 & \gamma_1^2 & \beta_1\gamma_1 & \gamma_1\alpha_1 & \alpha_1\beta_1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_5^2 & \beta_5^2 & \gamma_5^2 & \beta_5\gamma_5 & \gamma_5\alpha_5 & \alpha_5\beta_5 \end{vmatrix}=0.$$

44. *Concurrence of the straight line and conic.*

The well-known property that every right line meets a curve of the second degree in two real, coincident, or imaginary points* is readily exhibited.

Writing the general equations of the conic and straight line,

$$l\alpha+m\beta+n\gamma=0,$$

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0,$$

* Salmon's Conics, p. 132.

and the simultaneous relation

$$a\alpha + b\beta + c\gamma = 2\Delta,$$

we see from these three equations that we are enabled first to express β and γ as functions of α of the first degree, which substituted gives us a quadratic, and this in turn furnishes two roots, determining two points of intersection.

45. *Excursus upon the fundamental form of the equation to a conic section in trilinear coordinates.*

Conceive the vertex of a right circular cone placed at the origin O of x, y, z coordinates; XYZ the plane of section, and also the triangle of reference in trilinear coordinates; θ_1, θ_2, θ_3 the angles which the perpendicular upon this plane makes with OX, OY, OZ; OX and OY supposed to be at right angles to the axis of the cone OZ and to one another; P a point on the curve and the origin of α, β, γ.

Let a, b, c be the perpendicular distances of P from the coordinate planes, and d the diagonal from O in the lower face of the parallelepipedon.

Then the perpendiculars from P on $XY = \alpha$, on $XZ = \beta$, on $YZ = \gamma$.

By the geometry of the figure, a (the perpendicular distance of P from the plane OYZ) $= \alpha \sin \theta_1$, $b = \beta \sin \theta_2$, and c (the distance of P from the plane OXY) $= \gamma \sin \theta_3$,

$$d^2 = a^2 + b^2,$$

$$c = d \tan \theta,$$

where $\theta =$ semi-vertical angle of the cone.

Hence $$c^2 = (a^2 + b^2) \tan^2 \theta;$$

that is, $$\gamma^2 \sin^2 \theta_3 = (\alpha^2 \sin^2 \theta_1 + \beta^2 \sin^2 \theta_2) \tan^2 \theta,$$

or $$\alpha^2 \sin^2 \theta_1 \tan^2 \theta + \beta^2 \sin^2 \theta_2 \tan^2 \theta - \gamma^2 \sin^2 \theta_3 = 0,$$

which also may be written

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0,$$

where it is understood that the signs are not all the same.

We have, therefore, derived an equation to a conic section homogeneous and of the second degree in trilinear coordinates; and in turn it may easily be shown that the general equation

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0$$

may be made to take the form

$$l\alpha^2+m\beta^2+n\gamma^2=0;$$

and hence every equation of the second degree may be said to express some section of a right circular cone.

In the genesis of this equation it is evident how we might proceed to make some applications in tri-dimensional Geometry. For instance, let us take some function of x, y, z as an equation to a surface in three rectangular coordinates, as,

$$f(x, y, z)=0;$$

and let $$x\cos\theta_1+y\cos\theta_2+z\cos\theta_3=p$$

be the equation to any plane; also let the traces of the coordinate planes upon the plane of section be the lines of reference; then, if x, y, z be the coordinates of any point P upon the given surface, and if θ_1, θ_2, θ_3 be the angles which the proposed plane makes with the original plane, we must have

$$x=\alpha\sin\theta_1,\quad y=\beta\sin\theta_2,\quad z=\gamma\sin\theta_3;$$

and consequently, by substituting in the given surface, we obtain the trilinear equation to the section, that is,

$$f(\alpha\sin\theta_1,\ \beta\sin\theta_2,\ \gamma\sin\theta_3)=0.$$

In the same manner, the equation to the section of the

ellipsoid $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

by the same plane would evidently become

$$\frac{\alpha^2\sin^2\theta_1}{a^2}+\frac{\beta^2\sin^2\theta_2}{b^2}+\frac{\gamma^2\sin^2\theta_3}{c^2}=1 \quad\text{.........}\quad (1),$$

which is easily rendered homogeneous by first finding the identical relation among α, β, γ in the given plane; that is, by substituting the values of x, y, z as above when

$$\frac{\alpha \sin 2\theta_1 + \beta \sin 2\theta_2 + \gamma \sin 2\theta_3}{2p} = 1;$$

and consequently (1) becomes

$$\frac{\alpha^2 \sin^2 \theta_1}{a^2} + \frac{\beta^2 \sin^2 \theta_2}{b^2} + \frac{\gamma^2 \sin^2 \theta_3}{c^2} - \left[\frac{\alpha \sin 2\theta_1 + \beta \sin 2\theta_2 + \gamma \sin 2\theta_3}{2p}\right]^2 = 0.$$

46. By the last Article we are enabled at once to interpret such an equation as

$$\alpha\beta - k\gamma x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where, by ordinary abridged notation, $\alpha = 0$, $\beta = 0$, $\gamma = 0$, $x = 0$ are the equations to four straight lines, and k is any constant.

In considering the given equation, we see that it is of the second degree, and satisfied when $\alpha = 0$ and $\gamma = 0$ are at the same time satisfied; and hence we infer that the conic represented by the equation passes through the intersection of these lines. In the same manner, another point is determined by the intersection of $\beta = 0$ and $x = 0$, and so on for the four sets of lines determining four points through which the conic must pass; that is, (1) represents a conic circumscribing a quadrilateral whose sides are α, β, γ, and x.

From this we readily pass to the interpretation of the similar equation

$$\alpha\beta - k\gamma^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

which indicates that two of the opposite sides, γ and x, are coincident. And as each of the lines $\alpha = 0$ and $\beta = 0$ can meet the conic in but two points, they must be conceived as drawn from a point without, and hence as tangents to the conic at the points respectively where the coinciding lines meet the conic.*

* Salmon's Conics, p. 223.

47. *The triangle of reference self-conjugate with regard to the conic.*

Returning to the equation

$$l\alpha^2+m\beta^2+n\gamma^2=0,$$

we see that it expresses no possible locus while l, m, n are regarded as all positive or all negative; but, as we saw in Art. 45, these are not all of the same sign.

Let l, m be positive, n negative, and for them write u^2, v^2, w^2 respectively; then

$$u^2\alpha^2+v^2\beta^2-w^2\gamma^2=0 \text{ ....} (1),$$

or $$u^2\alpha^2+(v\beta+w\gamma)(v\beta-\gamma)=0.$$

After the analogy of equation (2) of Art. 46, the lines

$$v\beta+w\gamma=0,$$
$$v\beta-w\gamma=0,$$

must be tangents, and α their chord of contact; in other words, the line $\alpha=0$, which is the equation of BC, a side of the triangle of reference, is the chord of contact of a pair of tangents from the vertex A.

It is equally admissible to write (1)

$$v^2\beta^2+u^2\alpha^2-w^2\gamma^2=0,$$

whence we see, as before, $\beta=0$, which is the equation of AC, a side of the triangle of reference, is the chord of contact of the lines $$u\alpha+w\gamma=0,$$
and $$u\alpha-w\gamma=0,$$
which are tangents from the vertex B; or, still further, (1) may take the form

$$u^2\alpha^2+v^2\beta^2-w^2\gamma^2=0,$$

or $$(u\alpha+v\beta\sqrt{-1})(u\alpha-v\beta\sqrt{-1})-w^2\gamma^2=0;$$

whence $$u\alpha+v\beta\sqrt{-1}=0,$$
$$u\alpha-v\beta\sqrt{-1}=0,$$

are the imaginary tangents from the vertex C, and $\gamma=0$ their chord of contact, which is also the equation of AB.

Therefore, as we see, each side of the triangle of reference becomes in turn the chord of contact of tangents from the opposite angle, that is, the *polar* of that point with respect to the conic; and, conversely, each vertex is seen to be the *pole* of the opposite side, or the triangle may be described as self-conjugate with respect to the conic; which was to be shown.*

48. *Intercepts of a directed line upon the curve*

$$l\alpha^2+m\beta^2+n\gamma^2=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Let $(\alpha_1, \beta_1, \gamma_1)$ be the point from which the directed line h is drawn to meet the conic; s_1, s_2, s_3 sines of the given direction all measured in the same direction, the first from h to the parallel of BC, the second measured from the same point to the parallel of AC, and the third in the same direction round to the parallel of AB, of the sides, respectively, of the triangle of reference.

Then, evidently,

$$\alpha = \alpha_1+s_1h,$$
$$\beta = \beta_1+s_2h,$$
$$\gamma = \gamma_1+s_3h.$$

These values substituted in (1) give a quadratic in h; that is,

$$h^2(ls_1^2+ms_2^2+ns_3^2)+2h(ls_1\alpha_1+ms_2\beta_1+ns_3\gamma_1) + (l\alpha_1^2+m\beta_1^2+n\gamma_1^2)=0,$$

The two values of h obtained from this equation will be the lengths of the intercepts from the given point.

Suppose this point to be on the curve, we shall then have

$$l\alpha_1^2+m\beta_1^2+n\gamma_1^2=0,$$

and consequently but one value to h (one intercept becoming zero), which is manifestly the length of a chord in the given direction.

49. *Locus of middle points of parallel chords.*

Let the curve be the same as in the last Article; $(\alpha_1, \beta_1, \gamma_1)$

* Salmon's Conics, p. 227.

a point on the locus, and the chord from this point represented by h, whose direction is given by its sines, s_1, s_2, s_3.

Then, as before,

$$\alpha = \alpha_1 + s_1 h, \quad \beta = \beta_1 + s_2 h, \quad \gamma = \gamma_1 + s_3 h.$$

Hence the intercepts of the curve are given by

$$l(\alpha_1 + s_1 h)^2 + m(\beta_1 + s_2 h)^2 + n(\gamma_1 + s_3 h)^2 = 0,$$

which, by the supposition, are equal; that is, the two values of h will appear with opposite signs; and since they must be equal, their sum, or the coefficient of h, will be equal to 0, and consequently*

$$ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1 = 0,$$

a straight line giving the relation, in fact, of *any* point on the locus, and hence the equation required.

50. *Tangent to a point on the conic.*

Let now the point $(\alpha_1, \beta_1, \gamma_1)$ be on the conic. We have seen, by Art. 48, that when this point lies on the conic,

$$l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2 = 0;$$

and therefore the quadratic in h of that Article reduces to

$$(ls_1^2 + ms_2^2 + ns_3^2)\,h + 2\,(ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1) = 0,$$

which gives the length of the chord.

When, now, the direction becomes that of the tangent, the length of this chord, that is h, becomes zero, and we have

$$ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1 = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (1).$$

But if (α, β, γ) be any point on this tangent, we must have

$$s_1 = \frac{\alpha - \alpha_1}{h}, \quad s_2 = \frac{\beta - \beta_1}{h}, \quad s_3 = \frac{\gamma - \gamma_1}{h},$$

which values substituted in (1) give

$$l\alpha_1\alpha + m\beta_1\beta + n\gamma_1\gamma = l\alpha_1^2 + m\beta_1^2 + n\gamma_1^2 = 0.$$

* Bourdon's Algebra, p. 160.

Hence $$l\alpha_1\alpha + m\beta_1\beta + n\gamma_1\gamma = 0$$
expresses the required relation, and is therefore the equation sought.

51. *Coordinates of centre.*

The reasoning is similar to that of Art. 49. The direction of a diameter being s_1, s_2, s_3, the lengths of intercepts by the curve in this direction, measured from the centre $(\alpha_1, \beta_1, \gamma_1)$, will be given by the same equation as in that Article; and since the quadratic must, by the premises, give equal roots with opposite signs, the coefficient of h will $= 0$; that is,

$$ls_1\alpha_1 + ms_2\beta_1 + ns_3\gamma_1 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For the actual determination of α_1, β_1, γ_1 we have

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 2\Delta \quad\ldots\ldots\ldots\ldots\ldots\ldots (2);$$

but since $$a\alpha + b\beta + c\gamma = 2\Delta,$$

and $$\alpha = \alpha_1 + s_1 h, \quad \beta = \beta_1 + s_2 h, \quad \gamma = \gamma_1 + s_3 h,$$

we have $$as_1 + bs_2 + cs_3 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Comparing (1) and (3), we get

$$\frac{l\alpha_1}{a} = \frac{m\beta_1}{b} = \frac{n\gamma_1}{c}.$$

These ratios will enable us to find the values of α_1, β_1, γ_1; thus, by dividing (2) by $\frac{l\alpha_1}{a}$ or its equals, we have

$$\frac{a^2}{l} + \frac{b^2}{m} + \frac{c^2}{n} = \frac{2\Delta a}{l\alpha_1} = \frac{2\Delta b}{m\beta_1} = \frac{2\Delta c}{n\gamma_1};$$

and therefore the coordinates of the centre are determined.

It will be observed that these coordinates enable us to determine the condition that the conic may be a parabola; the centre of a parabola being infinitely distant, its coordinates must satisfy the relation

$$a\alpha_1 + b\beta_1 + c\gamma_1 = 0.$$

Making the substitution, we have the required condition,

that is, $$\frac{a^2}{l}+\frac{b^2}{m}+\frac{c^2}{n}=0.$$

52. *Equation of circle with respect to which the triangle of reference is self-conjugate.*

It may be inferred from Art. 47 that when an equation of the second degree does not involve $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, the conic, in such case, is so related to the triangle of reference that each side is the polar, with respect to this conic, of the opposite vertex.

Let one side, as CA, cut this conic in two points $(f_1, 0, h_1)$, $(f_2, 0, h_2)$, and let this chord be bisected. Then the equation of the straight line from the vertex to the point of bisection

is, evidently, $$\frac{\alpha}{f_1+f_2}=\frac{\gamma}{h_1+h_2} \quad\text{........................}\quad (1),$$

which line passes through the centre of the conic.

We have seen (Art. 47) that the conic may be written

$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2=0,$$

where, in this case, nothing is assumed as to which of the coefficients l, m, n should be attributed the negative sign.

Now f_1 and f_2 are identical with the values of α given by this equation.

We can eliminate β and γ by the relation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

remembering that $\beta=0$; and we have the quadratic

$$\alpha^2-\frac{4\Delta w^2a}{u^2c^2+w^2a^2}\alpha+\frac{4\Delta^2}{u^2c^2+w^2a^2}=0;$$

and therefore $$f_1+f_2=\frac{4\Delta w^2a}{u^2c^2+w^2a^2},$$

since this coefficient is the sum of the roots of α; by the same reasoning we have

$$h_1 + h_3 = \frac{4\Delta u^2 c}{w^2 a^2 + u^2 c^2},$$

and consequently (1) becomes

$$\frac{\alpha}{w^2 a} = \frac{\gamma}{u^2 c},$$

a line on which the centre lies, which may be written

$$\frac{u^2 \alpha}{a} = \frac{w^2 \gamma}{c}.$$

Similarly, $$\frac{u^2 \alpha}{a} = \frac{v^2 \beta}{b}.$$

Now it is a property of the circle that a line joining any point to the centre is perpendicular to the polar; therefore the line

$$\frac{u^2 \alpha}{a} - \frac{w^2 \gamma}{c} = 0,$$

which is drawn from the centre to the vertex B, is perpendicular to $\beta = 0$.

But, by the figure, we have

$$\frac{\alpha}{\cos C} = \frac{\gamma}{\cos A};$$

therefore $$\frac{u^2}{a \cos A} = \frac{w^2}{c \cos C};$$

similarly, $$\frac{u^2}{a \cos A} = \frac{v^2}{b \cos B};$$

and the equation of the circle becomes

$$a \cos A\, \alpha^2 + b \cos B\, \beta^2 + c \cos C\, \gamma^2 = 0,$$

or $$\sin 2A\, \alpha^2 + \sin 2B\, \beta^2 + \sin 2C\, \gamma^2 = 0.$$

The circle thus represented will be imaginary unless the triangle of reference have an obtuse angle.

53. *The inscribed triangle.*

Returning now to the general equation of the second degree,

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta = 0,$$

we see that if $\beta = 0,$

$\gamma = 0,$

the equation reduces to $u = 0.$

But this is the condition that the curve should pass through the vertex A. In the same manner, it may be shown that when $v = 0$ and $w = 0$ it will pass through B and C.

Under these conditions the equation reduces to

$$u_1\beta\gamma+v_1\gamma\alpha+w_1\alpha\beta = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

which also may now be written without the subscripts. We may therefore write it

$$u\beta\gamma+\alpha\,(v\gamma+w\beta) = 0;$$

and since every straight line cuts the curve in two points, the line $v\gamma+w\beta = 0$ must pass through the point where $\alpha = 0$ and $\beta = 0$, since these values alone will satisfy the equation; but these points are coincident, and determine the vertex A. This line could not therefore be drawn within the curve, for it would then meet it in three points; it must be drawn without, and therefore is the tangent at A.

Equation (1) is an equation of the second degree, and represents evidently, from what has been said, a curve circumscribing the triangle of reference, satisfied when any two coordinates $= 0$, in which case each vertex lies upon the locus.

54. The conic $u\beta\gamma+v\gamma\alpha+w\alpha\beta = 0$

will give values for the intercepts by the curve upon a straight line from a given point; the equation to the tangent at any point; the locus of middle points of parallel chords, in precisely the same manner as has already been shown in preceding Articles.

Let us here seek the condition that any straight line should be a tangent to the conic.

Since $$u\beta\gamma+v\gamma\alpha+w\alpha\beta=0$$

represents a conic described about the triangle of reference, it passes through the point, as we have seen, where $\beta=0$ and $\gamma=0$.

Let $$f\alpha+g\beta+h\gamma=0$$

be the straight line. If by means of this equation we eliminate α from the equation to the given conic, we must have, evidently, coincident values for $\beta:\gamma$.

Now the quadratic which results,

$$\frac{\beta^2}{\gamma^2}+\frac{\beta}{\gamma}\left(\frac{gv-fu+hw}{gw}\right)+\frac{vh}{gw}=0,$$

will give equal values for $\frac{\beta}{\gamma}$ when the value of the radical is zero; that is, when

$$4hgvw-(gv+hw-fu)^2=0,$$

or $$u^2f^2+v^2g^2+w^2h^2-2vwgh-2uwhf-2uvgf=0.$$

This may also be written in the form

$$\pm\sqrt{uf}\pm\sqrt{vg}\pm\sqrt{wh}=0,$$

which can be verified by clearing of radicals; and this is the condition that the straight line

$$f\alpha+g\beta+h\gamma=0$$

may touch the curve

$$u\beta\gamma+v\gamma\alpha+w\alpha\beta=0.$$

Dr. Salmon has called this the tangential equation of the curve.

55. *Pascal's hexagon*: the opposite sides of a hexagon inscribed in a conic meet, if produced, in collinear points.

Let the triangle of reference be inscribed in the curve, and let Ax_1, Bx_2, Cx_3 be three of the sides of the inscribed hexagon.

Since, if (f_1, g_1, h_1), (f_2, g_2, h_2), (f_3, g_3, h_3) be the coordinates of x_1, x_2, x_3 respectively, we shall have, by the figure,

$$\beta : \gamma :: g_1 : h_1;$$

Ax_1 will therefore be represented by

$$0, \quad h_1, \quad -g_1;$$

and x_2C by $$g_2, \quad -f_2, \quad 0.$$

Hence (Art. 12) the point of intersection of these sides is

$$g_1f_2, \quad g_1g_2, \quad h_1g_2.$$

The side Bx_2 will be subject to the coordinates f_2 and h_2; the side x_3A to h_3 and g_3; hence these sides will intersect in the point

$$f_2h_3, \quad h_2g_3, \quad h_3h_2.$$

The sides Cx_3 and x_1B will, in like manner, intersect in the point $$f_3f_1, \quad f_1g_3, \quad h_1f_3.$$

Hence we have, by the determinant of collinearity, the condition

$$\begin{vmatrix} g_1f_2 & g_1g_2 & h_1g_2 \\ f_2h_3 & h_2g_3 & h_3h_2 \\ f_3f_1 & f_1g_3 & h_1f_3 \end{vmatrix} = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(1)},$$

which, as is evident, is also the condition that the three points x_1, x_2, x_3 lie on one conic with the vertices of reference.

For let (f_1, g_1, h_1), (f_2, g_2, h_2), (f_3, g_3, h_3) be the three given points on the conic, and let the conic be represented by

$$ugh+vhf+wfg = 0.$$

Then will the three vertices of reference lie on this conic; and if the curve pass through the given points we must have

$$ug_1h_1+vh_1f_1+wf_1g_1 = 0,$$
$$ug_2h_2+vh_2f_2+wf_2g_2 = 0,$$
$$ug_3h_3+vh_3f_3+wf_3g_3 = 0;$$

and the determinant by which u, v, and w are eliminated is

$$\begin{vmatrix} g_1h_1 & h_1f_1 & f_1g_1 \\ g_2h_2 & h_2f_2 & f_2g_2 \\ g_3h_3 & h_3f_3 & f_3g_3 \end{vmatrix} = 0,$$

which is identical in result with the condition given in (1).

Exercises.

1. A triangle being inscribed in a conic, are the points collinear in which each side intersects the tangents at the opposite vertex?

2. Prove the theorem of Hermes, that if $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ be two points on the conic

$$u\beta\gamma+v\gamma\alpha+w\alpha\beta = 0,$$

then the equation to the straight line joining them is

$$\frac{u\alpha}{\alpha_1\alpha_2}+\frac{v\beta}{\beta_1\beta_2}+\frac{w\gamma}{\gamma_1\gamma_2} = 0.$$

3. When does $$u\beta\gamma + v\gamma\alpha + w\alpha\beta = 0$$
represent an hyperbola ?

4. What is the chord of contact of the tangents
$$u(\beta+\gamma) + (\sqrt{v} \pm \sqrt{w})^2 \alpha = 0 \,?$$

5. What is the condition of concurrence of the normals at the vertices of the triangle of reference to the above conic ?

CHAPTER IV.

POLE AND POLAR—RECIPROCATION.

56. *Inscribed Conic.*

Any conic inscribed in the triangle of reference may be represented by

$$\sqrt{l\alpha} + \sqrt{m\beta} + \sqrt{n\gamma} = 0,$$

which, cleared of radicals, is

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

as we have seen (Art. 54), where it expressed a particular condition.

If we examine this equation, we shall find that it may be written in each of the three following forms :

$$4mn\beta\gamma - (m\beta + n\gamma - l\alpha)^2 = 0 \text{ .... } (1),$$

$$4nl\,\alpha\gamma - (n\gamma + l\alpha - m\beta)^2 = 0 \text{ .... } (2),$$

$$4ml\,\alpha\beta - (l\alpha + m\beta - n\gamma)^2 = 0 \text{ .... } (3),$$

from which, as they differ only by a constant from the equation interpreted in Art. 54, we conclude from parallel reasoning that each represents a conic section in which the factors of the first terms equated separately to zero are tangents to the curve in whose equation they respectively appear, and the second terms are the squares of their respective chords of contact.

Hence the lines of reference are tangents, and the conic is an inscribed conic.

57. *Conversely, every conic whose lines of reference are the sides of a circumscribed triangle will have an equation of the form*

$$\pm\sqrt{l\alpha} \pm \sqrt{m\beta} \pm \sqrt{n\gamma} = 0,$$

since every conic may be represented by

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0.$$

If the triangle of reference be circumscribed, the side BC will be a tangent and be represented by $\alpha=0$. This value substituted in the general equation gives

$$v\beta^2+2u_1\beta\gamma+w\gamma^2=0,$$

which, from the nature of the case, must have equal roots, that is, the left-hand member of the equation must be a perfect square; hence $\qquad u_1^2=vw;$

that is, $\qquad u_1=\pm\sqrt{vw};$

and similarly $\qquad v_1=\pm\sqrt{wu},$

$$w_1=\pm\sqrt{uv},$$

are the necessary and sufficient conditions that the conic should touch the lines $\beta=0$ and $\gamma=0$.

Substituting these values in the general equation, and remembering to write l^2, m^2, n^2 for u, v, w, we have

$$\pm\sqrt{l\alpha}\pm\sqrt{m\beta}\pm\sqrt{n\gamma}=0\ \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

which was to be proved.

58. Four conics may be inscribed in the triangle of reference so related that the points of contact shall lie on the lines represented by $\qquad \pm l\alpha\pm m\beta\pm n\gamma=0.$

For it is evident that (1) of the last Article may be written

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2\pm 2mn\beta\gamma\pm 2nl\gamma\alpha\pm 2lm\alpha\beta=0,$$

which, writing all the doubtful signs negative, or one negative only at a time, breaks up into the equations to four conics, and we are presented with four interpretations similar to (1), (2), (3) of Art. 56. If the double signs be taken otherwise, the locus will become simply two coincident straight lines.

These equations therefore, as representing conics, have twelve points of contact lying three and three on the above four straight lines. It may be observed that the actual sign of the quantities under the radicals in equation (1) of the last Article depends upon which sign is taken with the coefficients of $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$.

The process for finding tangent, intercepts, centre of conic, &c. is similar to that already exhibited in the last Chapter, and need not be repeated.

59. *Brianchon's Hexagon: the three opposite diagonals of every hexagon described about a conic concur.*

The method of proof is quite similar to that already exhibited. Let three sides be produced for the triangle of reference; $ABCDEF$ the hexagon; AB, CD, EF the sides produced.

If
$$l_1\alpha+m_1\beta+n_1\gamma=0$$
be the equation to AF,
$$l_2\alpha+m_2\beta+n_2\gamma=0$$
to that of BC, and
$$l_3\alpha+m_3\beta+n_3\gamma=0$$
to that of DE; then the diagonals AD and FC will be represented as follows:—

The point A, $\gamma=0$, and $l_1\alpha+m_1\beta=0$;

" D, $\alpha=0$, and $m_3\beta+n_3\gamma=0$;

(AD), $$l_1m_3\alpha+m_1m_3\beta+m_1n_3\gamma=0;$$

(FC), $$l_1n_2\alpha+n_1m_2\beta+n_1n_2\gamma=0;$$

(BE), $$l_2l_3\alpha+l_3m_2\beta+l_2n_3\gamma=0.$$

Hence, (Art. 8),
$$\begin{vmatrix} l_1m_3 & m_1m_3 & m_1n_3 \\ l_1n_2 & n_1m_2 & n_1n_2 \\ l_2l_3 & l_3m_2 & l_2n_3 \end{vmatrix}=0.$$

The condition that the three lines AF, BC, and DE shall touch the conic

$$\sqrt{l\alpha}+\sqrt{m\beta}+\sqrt{n\gamma}=0$$

is found by first finding the condition of tangency of each of these lines, which is, for AF,

$$\frac{l}{l_1}+\frac{m}{m_1}+\frac{n}{n_1}=0;$$

for BC,
$$\frac{l}{l_2}+\frac{m}{m_2}+\frac{n}{n_2}=0;$$

for DE,
$$\frac{l}{l_3}+\frac{m}{m_3}+\frac{n}{n_3}=0;$$

and therefore
$$\begin{vmatrix} \frac{1}{l_1} & \frac{1}{m_1} & \frac{\,}{n_1} \\ \frac{1}{l_2} & \frac{1}{m_2} & \frac{1}{n_2} \\ \frac{1}{l_3} & \frac{1}{m_3} & \frac{1}{n_3} \end{vmatrix}=0,$$

which, it is seen, is the same condition as above.

From what follows on reciprocation it will be evident that, by reciprocating Pascal's Theorem, Brianchon's Theorem may be obtained.

60. It is proper here to notice a different form of notation which is frequently employed in this subject.

Suppose $f(\alpha,\beta,\gamma)=0$ to represent the equation to the curve, and s_1, s_2, s_3 the direction-sines of the tangent at the point $(\alpha_1,\beta_1,\gamma_1)$; (α,β,γ) any point on the tangent; and h the distance between these two points. Then, following the reasoning of Art. 48, the intercepts on h will be given by substituting the new values,

$$\alpha=\alpha_1+s_1h,\quad \beta=\beta_1+s_2h,\quad \gamma=\gamma_1+s_3h,$$

in the above equation; that is, by

$$f(\alpha_1+s_1h,\ \beta_1+s_2h,\ \gamma_1+s_3h) = 0,$$

which, when expanded as we have already seen in the Article referred to, will consist of some function of $(\alpha_1, \beta_1, \gamma_1)$, a coefficient of h, and a coefficient of h^2 which is some function of the direction-sines, or

$$f(\alpha_1, \beta_1, \gamma_1) + h^2 f(s_1, s_2, s_3) + h\left(s_1\frac{df}{d\alpha_1} + s_2\frac{df}{d\beta_1} + s_3\frac{df}{d\gamma_1}\right) = 0.$$

If now $(\alpha_1, \beta_1, \gamma_1)$ lie on the curve, we must have

$$f(\alpha_1, \beta_1, \gamma_1) = 0;$$

also one of the intercepts becomes zero, and since the line is a tangent the length of the chord is zero, that is, the coefficient of h vanishes, and we have

$$s_1\frac{df}{d\alpha_1} + s_2\frac{df}{d\beta_1} + s_3\frac{df}{d\gamma_1} = 0.$$

By substituting for s_1, s_2, s_3 their values, we shall obtain twice the function in $(\alpha_1, \beta_1, \gamma_1)$ which $= 0$, that is,

$$\alpha_1\frac{df}{d\alpha_1} + \beta_1\frac{df}{d\beta_1} + \gamma_1\frac{df}{d\gamma_1} = 2f(\alpha_1\beta_1\gamma_1),$$

as may be shown by taking the differential coefficients of

$$u\alpha_1^2 + v\beta_1^2 + w\gamma_1^2 + 2u_1\beta_1\gamma_1 + 2v_1\gamma_1\alpha_1 + 2w_1\alpha_1\beta_1 = 0,$$

in respect to $\alpha_1, \beta_1, \gamma_1$ respectively, multiplying the differential coefficients by each of these coordinates and adding, when we shall find that

$$2f(\alpha_1, \beta_1, \gamma_1) = 0,$$

since $(\alpha_1, \beta_1, \gamma_1)$ is supposed to be on the curve. There will remain, therefore,

$$\alpha\frac{df}{d\alpha_1} + \beta\frac{df}{d\beta_1} + \gamma\frac{df}{d\gamma_1} = 0,$$

the equation to the tangent at $(\alpha_1, \beta_1, \gamma_1)$, since it expresses a relation among the coordinates of any point in the line.

61. *Polar of a point in respect to the conic.*

Let the fixed point be $(\alpha_1, \beta_1, \gamma_1)$; $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$ the coordinates of points of contact of tangents from the given point. Then we can show, by an extension of the reasoning of the last Article, that

$$\alpha_2 \frac{df}{d\alpha_1} + \beta_2 \frac{df}{d\alpha_1} + \gamma_2 \frac{df}{d\alpha_1} = 0$$

is the tangent from $(\alpha_1, \beta_1, \gamma_1)$ to the point of contact $(\alpha_2, \beta_2, \gamma_2)$;

and likewise $$\alpha_3 \frac{df}{d\alpha_1} + \beta_3 \frac{df}{d\beta_1} + \gamma_3 \frac{df}{d\gamma_1} = 0$$

is the equation to the tangent at $(\alpha_3, \beta_3, \gamma_3)$. Therefore these equations express the fact that the line joining these points of contact is a locus whose equation is

$$\alpha \frac{df}{d\alpha_1} + \beta \frac{df}{d\beta_1} + \gamma \frac{df}{d\gamma_1} = 0,$$

that is, the polar with respect to the conic

$$f(\alpha, \beta, \gamma) = 0;$$

or we may proceed otherwise. Defining the polar of a given point as the locus of the intersection of tangents drawn to the points of section by a straight line through the given point, we should have for the equation through the three points in the same straight line

$$\alpha(\beta_1\gamma_2 - \beta_2\gamma_1) + \beta(\gamma_1\alpha_2 - \gamma_2\alpha_1) + \gamma(\alpha_1\beta_2 - \alpha_2\beta_1) = 0 \ldots (1),$$

where (α, β, γ) is the given point, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_1, \beta_1, \gamma_1)$ the points of section in which any straight line cuts the conic.

The intersection of tangents,

$$\alpha_1 \frac{df}{d\alpha} + \beta_1 \frac{df}{d\beta} + \gamma_1 \frac{df}{d\gamma} = 0,$$

and

$$\alpha_2 \frac{df}{d\alpha} + \beta_2 \frac{df}{d\beta} + \gamma_2 \frac{df}{d\gamma} = 0,$$

will be

$$\frac{\frac{df}{d\alpha}}{\beta_1\gamma_2 - \beta_2\gamma_1} = \frac{\frac{df}{d\beta}}{\gamma_1\alpha_2 - \gamma_2\alpha_1} = \frac{\frac{df}{d\gamma}}{\alpha_1\beta_2 - \alpha_2\beta_1} \quad \ldots\ldots\ldots \; (2).$$

Equation (1) with (2) gives

$$\alpha \frac{df}{d\alpha} + \beta \frac{df}{d\beta} + \gamma \frac{df}{d\gamma} = 0.$$

This equation being independent of $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ is the relation at the intersection of the tangents; it is therefore the locus required, and, by definition, the polar of (α, β, γ).

62. *Coordinates of the pole of a straight line in respect to a conic.*

Let

$$f\alpha + g\beta + h\gamma = 0$$

be the equation to the straight line, and

$$\phi\,(\alpha, \beta, \gamma) = 0$$

to that of the conic. If (α, β, γ) be the coordinates of the required point, then its polar, by the last Article, is

$$\alpha \frac{d\phi}{d\alpha} + \beta \frac{d\phi}{d\beta} + \gamma \frac{d\phi}{d\gamma} = 0;$$

and since this is the same as the given line, we have

$$\frac{\frac{d\phi}{d\alpha}}{f} = \frac{\frac{d\phi}{d\beta}}{g} = \frac{\frac{d\phi}{d\gamma}}{h};$$

that is,

$$\frac{u\alpha + w_1\beta + v_1\gamma}{f} = \frac{v\beta + u_1\gamma + w_1\alpha}{g} = \frac{w\gamma + v_1\alpha + u_1\beta}{h}.$$

Putting each member $= -s$, we have

$$u\alpha + w_1\beta + v_1\gamma + sf = 0,$$
$$v\beta + u_1\gamma + w_1\alpha + sg = 0,$$
$$w\gamma + v_1\alpha + u_1\beta + sh = 0\,;$$

and consequently

$$\frac{\alpha}{\begin{vmatrix} f & g & h \\ v_1 & u_1 & w \\ w_1 & v & u_1 \end{vmatrix}} = \frac{\beta}{\begin{vmatrix} f & g & h \\ u & w_1 & v_1 \\ v_1 & u_1 & w \end{vmatrix}} = \frac{\gamma}{\begin{vmatrix} f & g & h \\ w_1 & v & u_1 \\ u & w_1 & v_1 \end{vmatrix}}$$

which, with $$a\alpha + b\beta + c\gamma = 2\Delta,$$

determine the coordinates required.

63. *Centre of conic.*

If the equation be $\phi(\alpha, \beta, \gamma) = 0$,

and $(\alpha_1, \beta_1, \gamma_1)$ be the centre; then the roots of

$$\phi(\alpha_1 + s_1 h, \beta_1 + s_2 h, \gamma_1 + s_3 h) = 0$$

will be equal and opposite in sign.

Hence, since the coefficient of h must vanish in the quadratic,

therefore $$s_1 \frac{d\phi}{d\alpha_1} + s_2 \frac{d\phi}{d\beta_1} + s_3 \frac{d\phi}{d\gamma_1} = 0.$$

Bearing in mind the proportionality of s_1, s_2, s_3, that is, the relation $$as_1 + bs_2 + cs_3 = 0,$$

we have $$\frac{\dfrac{d\phi}{d\alpha_1}}{a} = \frac{\dfrac{d\phi}{d\beta_1}}{b} = \frac{\dfrac{d\phi}{d\gamma_1}}{c},$$

which, fully written out, will give determinants similar in form to those of the last Article, with a, b, c in place of f, g, h, and $\alpha_1, \beta_1, \gamma_1$ in place of α, β, γ.

64. *The conic will break up into two right lines* when we have the condition

$$\begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} = 0.$$

For suppose the two lines into which

$$\phi(\alpha, \beta, \gamma) = 0$$

breaks up to be represented by

$$f\alpha + g\beta + h\gamma = 0,$$
$$f_1\alpha + g_1\beta + h_1\gamma = 0.$$

Then will

$$\phi(\alpha, \beta, \gamma) = (f\alpha + g\beta + h\gamma)(f_1\alpha + g_1\beta + h_1\gamma),$$

and $$\frac{d\phi}{d\alpha} = f(f_1\alpha + g_1\beta + h_1\gamma) + f_1(f\alpha + g\beta + h\gamma),$$

with corresponding values for $\frac{d\phi}{d\beta}$ and $\frac{d\phi}{d\gamma}$. Hence, reverting to a principle already explained (Art. 31),

$$\frac{d\phi}{d\alpha} = 0, \quad \frac{d\phi}{d\beta} = 0, \quad \frac{d\phi}{d\gamma} = 0,$$

are straight lines which pass through the intersection of the given lines; that is, the lines

$$u\alpha + w_1\beta + v_1\gamma = 0,$$
$$w_1\alpha + v\beta + u_1\gamma = 0,$$
$$v_1\alpha + u_1\beta + w\gamma = 0,$$

concur, and give the above determinant.

65. When some of the four points of intersection of two conics become coincident, some of the common chords will

coincide; others will touch at a common point, that is, become tangents. There will, in general, be three pairs of common chords; if two points coincide, the conics touch; if the two remaining points also coincide, the conics have double contact and a chord of contact.

66. *Equation to the asymptotes.*

From Art. 64 we can easily form the equation to any pair of common chords. Thus, if

$$\phi(\alpha, \beta, \gamma) = Q\alpha^2 + R\beta^2 + S\gamma^2 + 2Q_1\beta\gamma + 2R_1\gamma\alpha + 2S_1\alpha\beta = 0$$

and

$$\phi_1(\alpha, \beta, \gamma) = u\alpha^2 + v\beta^2 + w\gamma^2 + 2u_1\beta\gamma + 2v_1\gamma\alpha + 2w_1\alpha\beta = 0$$

represent the two conics, the locus in question will have the equation (Art. 31),

$$\phi(\alpha, \beta, \gamma) + k\phi_1(\alpha, \beta, \gamma) = 0 \quad \ldots\ldots\ldots\ldots\ldots(1),$$

which must be so conditioned in k as to represent two straight lines, hence (Art. 64) *

$$\begin{vmatrix} Q + ku & S_1 + kw_1 & R_1 + kv_1 \\ S_1 + kw_1 & R + kv & Q_1 + ku_1 \\ R_1 + kv_1 & Q_1 + ku_1 & S + kw \end{vmatrix} = 0.$$

If now $\phi(\alpha, \beta, \gamma) = 0$ breaks up into two coincident straight lines, as,

$$(f\alpha + g\beta + h\gamma)^2 = 0,$$

we shall find

$$k = \frac{\begin{vmatrix} u & w_1 & v_1 & f \\ w_1 & v & u_1 & g \\ v_1 & u_1 & w & h \\ f & g & h & 0 \end{vmatrix} (=U),}{\begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} (=W),}$$

* Ferrers, p. 85.

which, substituted in (1), gives

$$\phi(\alpha, \beta, \gamma) W + U\phi_1(\alpha, \beta, \gamma) = 0,$$

or $$(f\alpha + g\beta + h\gamma)^2 W + U\phi_1(\alpha, \beta, \gamma) = 0 \quad \text{.........(2)}.$$

This equation now represents, under the above condition, not a pair of common chords, but a pair of common tangents whose chord of contact is

$$f\alpha + g\beta + h\gamma = 0.$$

We have now only to introduce the condition that the chord of contact is at infinity; that is, that

$$a\alpha + b\beta + c\gamma = 0;$$

wherefore (2) becomes

$$(a\alpha + b\beta + c\gamma)^2 \begin{vmatrix} u & w_1 & v_1 \\ w_1 & v & u_1 \\ v_1 & u_1 & w \end{vmatrix} + \phi_1(\alpha, \beta, \gamma) \begin{vmatrix} u & w_1 & v_1 & a \\ w_1 & v & u_1 & b \\ v_1 & u_1 & w & c \\ a & b & c & 0 \end{vmatrix} = 0,$$

which is the equation of the asymptotes.

Cor. 1.—Since every parabola has one tangent altogether at an infinite distance, the vanishing of the second determinant in the above equation expresses the condition that the conic may be a parabola.*

Cor. 2.—The conic will be a rectangular hyperbola when the asymptotes are at right angles to one another; that is, when (Art. 14) the two straight lines into which the conic breaks are subject to the condition of perpendicularity,

$$ll_1 - (mn_1 + m_1 n)\cos A + mm_1 - (nl_1 + n_1 l)\cos B$$
$$+ nn_1 - (lm_1 + l_1 m)\cos C.$$

In other words, if the conic be

$$\phi_1(\alpha, \beta, \gamma) = 0,$$

* Salmon's Conics, p. 224.

the required condition becomes

$$u+v+w-2u_1\cos A-2v_1\cos B-2w_1\cos C=0.$$

DEF. — W is called the discriminant of the function $\phi_1(\alpha, \beta, \gamma)$, and U the bordered discriminant of the same function (D. 43), where f, g, h, as the coefficients of α, β, γ, are $=\frac{a}{2\Delta}, \frac{b}{2\Delta}, \frac{c}{2\Delta}$ respectively. For the conic,

$$l\alpha^2+m\beta^2+n\gamma^2=0,$$

$$W=lmn,$$

$$U=-(a^2mn+b^2nl+c^2lm)\frac{1}{4\Delta^2}.$$

Numerous other functions may be determined.

67. Space does not permit extended illustration of the use of the abridged notation thus far exhibited. The reader can easily apply it. For instance, if the function be $\phi_1(\alpha, \beta, \gamma)$, and we wish to express the equation of the straight line at infinity in terms of the derived functions, the required equation might be written

$$f\frac{d\phi_1}{d\alpha}+g\frac{d_1}{d\beta}+h\frac{d\phi_1}{d\gamma}=0;$$

and since $$a\alpha+b\beta+c\gamma=0 \quad\text{.........................(1)}$$

represents the straight line at infinity, we have

$$\frac{uf+w_1g+v_1h}{a}=\frac{w_1f+vg+u_1h}{b}=\frac{v_1f+u_1g+wh}{c}.$$

These equivalents represented by $-k$ give us

$$uf+w_1g+v_1h+ak=0 \quad\text{..................(2)},$$

$$w_1f+vg+u_1h+bk=0 \quad\text{..................(3)},$$

$$v_1f+u_1g+wh+ck=0 \quad\text{..................(4)}.$$

Eliminating now between (1), (2), (3), (4), we obtain the condition that the minors of the bordered discriminant in respect to its f, g, h are proportional to f, g, h of the given equation, which minors being represented by A, B, C, the equation becomes

$$A\frac{d\phi_1}{d\alpha}+B\frac{d\phi_1}{d\beta}+C\frac{d\phi_1}{d\gamma}=0$$

as the straight line at infinity.

68. *The equation of the nine-point circle.*

We first find the condition that the conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u_1\beta\gamma+2v_1\gamma\alpha+2w_1\alpha\beta=0$$

may represent a circle. If the conic be a circle, $f(s_1, s_2, s_3)$ is constant, that is, all diameters will be equal; and since, in the equation for finding the lengths of the intercepts,

$$f(\alpha, \beta, \gamma)+h\left(s_1\frac{df}{d\alpha}+s_2\frac{df}{d\beta}+s_3\frac{df}{d\gamma}\right)+h^2f(s_1, s_2, s_3)=0,$$

the coefficient of h vanishes, we have

$$f(\alpha, \beta, \gamma)+h^2f(s_1, s_2, s_3)=0,$$

which gives the radius in the given direction. To reduce this, we may express the condition that diameters in three directions (that is, directions of the lines of reference) are equal.

We have, therefore, to express this,

$$h^2=\frac{f(\alpha, \beta, \gamma)}{f(x)}=\frac{f(\alpha, \beta, \gamma)}{f(y)}=\frac{f(\alpha, \beta, \gamma)}{f(z)}.$$

Wherefore $\qquad f(x)=f(y)=f(z);$

or, for direction of BC,

$$s_1=0, \quad s_2=\sin C, \quad s_3=-\sin B.$$

Hence $\qquad f(x)=f(0, c, -b).$

Similarly, for CA and AB,

$$f(y) = f(-c, \ 0, \ a),$$
$$f(z) = f(\ b, \ -a, 0),$$

from the proportionality of $\sin A$, $\sin B$, $\sin C$.

Hence we have the two conditions,

$$vc^2 + wb^2 - 2u_1 bc = wa^2 + uc^2 - 2v_1 ca = ub^2 + va^2 - 2w_1 ab.$$

In the second place, we see that, if the curve pass through the middle points of the sides of reference, α, β, γ must in succession be taken $= 0$; whence

$$\left.\begin{aligned} vc^2 + wb^2 + 2u_1 bc &= 0 \\ wa^2 + uc^2 + 2v_1 ca &= 0 \\ ub^2 + va^2 + 2w_1 ab &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1)},$$

which follows from the condition involved, that

$$b\beta = c\gamma = a\alpha.$$

Comparing the two sets of equations, we find

$$w_1 ab = v_1 ca = u_1 bc.$$

Hence, if equations (1) are true, they will hold whatever the value of u_1. Let $u_1 = -a$.

The resolution of these equations gives

$$u = 2a \cos A,$$
$$v = 2b \cos B,$$
$$w = 2c \cos C.$$

Hence the circle which passes through the middle points of the sides of reference (*the nine-point circle*) becomes

$$\alpha^2 \sin A \cos A + \beta^2 \sin B \cos B + \gamma^2 \sin C \cos C$$
$$- \beta\gamma \sin A - \gamma\alpha \sin B - \alpha\beta \sin C = 0,$$

or $\alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$

$$- 2\beta\gamma \sin A - 2\gamma\alpha \sin B - 2\alpha\beta \sin C = 0.$$

COR. 1.—If now $\alpha = 0$,

$$\beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma \sin A = 0.$$

But since $\qquad 2 \sin A = 2 \sin (B+C),$

$$\beta^2 \sin 2B + \gamma^2 \sin 2C - 2\beta\gamma (\sin B \cos C + \sin C \cos B) = 0.$$

This breaks up into the factors

$$(\beta \sin B - \gamma \sin C)(\beta \cos B - \gamma \cos C) = 0.$$

The circle therefore meets BC in two points.

The one when, by the hypothesis,

$$\alpha = 0, \quad b\beta = c\gamma, \quad i.\,e., \ \beta \sin B = \gamma \sin C,$$

which determines the middle of BC.

The other is evidently when

$$\alpha = 0, \quad \beta \cos B = \gamma \cos C,$$

the foot of the perpendicular from A. Similarly for the other sides.

COR. 2.—The last equation of this Article shows that the nine-point circle passes through the points of intersection of the circumscribed circle and the circle in respect to which the triangle of reference is self-conjugate.

COR. 3.—The difference between the equations of the circumscribed circle and the circle through the middle points of the sides of the triangle is

$\alpha \cos A + \beta \cos B + \gamma \cos C = 0$ multiplied by a constant,

since $\qquad \alpha^2 \sin 2A + \beta^2 \sin 2B + \gamma^2 \sin 2C$

$$= (\alpha \cos A + \beta \cos B + \gamma \cos C)(\alpha \sin A + \beta \sin B + \gamma \sin C).$$

But $\alpha \sin A + \beta \sin B + \gamma \sin C$ is a constant, and therefore $\alpha \cos A + \beta \cos B + \gamma \cos C = 0$ is their radical axis, or the homological axis of the triangle of reference and that formed by joining the feet of the perpendiculars.

COR. 4. — By similar reasoning we find that the same circle passes through the middle points of the sides of the triangles of which the point of intersection of perpendiculars is the vertex. Nine points are therefore determined.

POLAR RECIPROCALS.

69. Reciprocation—the principle of duality, or that analysis (or synthesis) which, while determining the distribution of points, coordinately fixes the position of lines—though of great interest, is altogether too large a subject for this Tract. Some theorems may be introduced. In general, we may say that to reciprocate involves interchanging "angular points" for "sides," "inscribing" for "circumscribing," "join" for "intersect," &c. &c.

For instance, if the well-known theorem, that "If two triangles be inscribed in a conic, their sides will be tangent to a conic," be reciprocated, we may write, "If two triangles circumscribe one conic, their vertices will lie on a conic."

This proof and its reciprocal may be exhibited by a common process in triangular and tangential coordinates (Arts. 24, 27). Let vertices of one triangle (sides of the same) be represented by (p_1, q_1, r_1), (p_2, q_2, r_2), (p_3, q_3, r_3); let the other be the triangle of reference, and suppose

$$f(p, q, r) = 0$$

the tangential equation of the conic passing through the points of reference; or the equation may be represented in both systems by

$$lqr + mpr + npq = 0.$$

Then the equations to the one triangle will be

$$\frac{lp}{p_2 p_3} + \frac{mq}{q_2 q_3} + \frac{nr}{r_2 r_3} = 0,$$

$$\frac{lp}{p_3 p_1} + \frac{mq}{q_3 q_1} + \frac{nr}{r_3 r_1} = 0,$$

$$\frac{lp}{p_1 p_2} + \frac{mq}{q_1 q_2} + \frac{nr}{r_1 r_2} = 0.$$

By comparing (Arts. 56, 57), we see that by this form of representation the inscribed conic (circumscribing) may be expressed by

$$\sqrt{Lp}+\sqrt{Mq}+\sqrt{Nr}=0;$$

that is, this conic will be inscribed in (circumscribe) both triangles provided the conditions of tangency be satisfied,

$$\left.\begin{array}{l}\dfrac{Lp_2p_3}{l}+\dfrac{Mq_2q_3}{m}+\dfrac{Nr_2r_3}{n}=0\\[2ex] \dfrac{Lp_3p_1}{l}+\dfrac{Mq_3q_1}{m}+\dfrac{Nr_3r_1}{n}=0\\[2ex] \dfrac{Lp_1p_2}{l}+\dfrac{Mq_1q_2}{m}+\dfrac{Nr_1r_2}{n}=0\end{array}\right\} \quad \ldots\ldots\ldots\ldots (1).$$

But since the given points (p_1, q_1, r_1, &c.), (vertices), lie by hypothesis on the conic, the condition must be expressed by

$$\begin{vmatrix}\dfrac{1}{p_1} & \dfrac{1}{q_1} & \dfrac{1}{r_1}\\[2ex] \dfrac{1}{p_2} & \dfrac{1}{q_2} & \dfrac{1}{r_2}\\[2ex] \dfrac{1}{p_3} & \dfrac{1}{q_3} & \dfrac{1}{r_3}\end{vmatrix}=0,$$

which condition satisfies equations (1), and proves the theorem and its reciprocal. The determinant follows, it is evident, as the eliminant of the equations,

$$\frac{l}{p_1}+\frac{m}{q_1}+\frac{n}{r_1}=0,$$

$$\frac{l}{p_2}+\frac{m}{q_2}+\frac{n}{r_2}=0,$$

$$\frac{l}{p_3}+\frac{m}{q_3}+\frac{n}{r_3}=0.$$

70. If m, n, p, q are the poles of the sides of a polygon *abcd*, then the points a, b, c, d are the poles of the sides of the polygon *mnpq*.

The conic with respect to which the poles and polars are taken is the auxiliary conic.

The reciprocal of a conic is a conic.

By Art. 60, the polar is given by

$$\alpha \frac{d\phi}{d\alpha} + \beta \frac{d\phi}{d\beta} + \gamma \frac{d\phi}{d\gamma} = 0.$$

If therefore (f, g, h) be any point on the reciprocal curve, its polar with respect to the auxiliary conic,

$$U\alpha^2 + V\beta^2 + W\gamma^2 = 0 \quad \dots\dots\dots\dots (1),$$

will be given by the equation

$$Uf\alpha + Vg\beta + Wh\gamma = 0 \quad \dots\dots\dots\dots (2).$$

Let the conic to be reciprocated be

$$l\alpha^2 + m\beta^2 + n\gamma^2 = 0 \quad \dots\dots\dots\dots (3).$$

To find the condition that (2) may touch (3), we eliminate α between the equation of the conic and the line; and if the line be a tangent, the values of $\beta : \gamma$ must be equal (Art. 57), and we obtain

$$\frac{U^2f^2}{l} + \frac{V^2g^2}{m} + \frac{W^2h^2}{n} = 0.$$

This being of the second degree, giving two points of intersection of the straight line, is a conic, and is the reciprocal of (3) with respect to (1).

71. Two straight lines are conjugate when each passes through the pole of the other. Required to express this condition. Let

$$f_1\alpha + g_1\beta + h_1\gamma = 0,$$

$$f_2\alpha + g_2\beta + h_2\gamma = 0,$$

be the two lines. Then (Art. 62) we may express the condition by the equation

$$\begin{vmatrix} \dfrac{d^2\phi}{d\alpha^2} & \dfrac{d^2\phi}{d\alpha\, d\beta} & \dfrac{d^2\phi}{d\alpha\, d\gamma} & f_1 \\ \dfrac{d^2\phi}{d\beta\, d\alpha} & \dfrac{d^2\phi}{d\beta^2} & \dfrac{d^2\phi}{d\beta\, d\gamma} & g_1 \\ \dfrac{d^2\phi}{d\gamma\, d\alpha} & \dfrac{d^2\phi}{d\gamma\, d\beta} & \dfrac{d^2\phi}{d\gamma^2} & h_1 \\ f_2 & g_2 & h_2 & 0 \end{vmatrix} = 0,$$

where $\phi\,(\alpha, \beta, \gamma) = u\alpha^2 + v\beta^2 + w\gamma^2 + \&c.$

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