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Wright, W. J.
Tracts ...

MATHEMATICAL TRACTS.

No. I.

DETERMINANTS.

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Geo. P. Davidson

June 14/77

TRACTS

RELATING TO THE

MODERN HIGHER MATHEMATICS.

TRACT No. 1.

DETERMINANTS.

BY

REV. W. J. WRIGHT, A.M.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY.

"That vast theory, transcendental in point of difficulty, elementary in regard to its being the basis of researches in the higher arithmetic, and in analytical geometry." —(M. HERMITE, quoted by Prof. SYLVESTER in *Phil. Mag.* 1852.)

LONDON:
C. F. HODGSON & SON, GOUGH SQUARE,
FLEET STREET.

1875.

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My acknowledgments are due to R. Tucker, Esq., M.A., Honorary Secretary of the London Mathematical Society, for valuable assistance rendered in passing these sheets through the press.—W. J. W.

Davidson

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CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

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INTRODUCTION TO THE SERIES.

DURING some intervals of foreign travel, and consequent interruption of formal ministerial labor, I resolved to begin the preparation of a Series of Elementary Tracts upon the following subjects in the Modern Higher Mathematics; viz—

Trilinear Coordinates.
Invariants.
Theory of Surfaces.
Elliptic Integrals.
Quaternions.

Upon further reflection, I have concluded to introduce the Series by a treatise upon Determinants, brief and very elementary, but sufficiently inclusive and rigorous to support and explain the references to this theory which are involved in the ordinary exposition of the first three subjects of the proposed list.

In undertaking this labor, I hope to turn the attention of the students of my country, especially those who are desirous of becoming Mathematicians, to these studies, which at present lie considerably beyond the usual "Scientific Course," even in our best colleges, but which the demands of Physics and higher Engineering must soon

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bring within it. I purpose, therefore, to give a strictly elementary view of the principal developments of the Pure Mathematics since the year 1841. I mark this year, not only because it is the proper initial point at which to begin the proposed survey, but because the year itself was remarkably rich in mathematical productions.

Jacobi, in that year, exhibited the versatility of his genius, whose power twelve years before had been proved by his "*Nova Fundamenta,*" in giving to the world his celebrated Memoirs "*De formatione et proprietatibus Determinantium*" and *De Determinantibus Functionalibus,*" which have been the bases of all subsequent labors in the Theory of Determinants.

In the same year, Dr. Geo. Boole laid down the principles out of which has grown the Modern Higher Algebra.

In the year 1841, also, was published, in the seventh volume of "*Mémoires des Savans étrangers,*" the full text of the general theory of the Abelian Functions, although what was known as Abel's Theorem had appeared much earlier.

Before the close of that year, last and perhaps least, but confessedly of immense influence on the British Universities, was published Gregory's "Processes and Examples of the Differential and Integral Calculus."

At this period Modern Geometry was unknown. Indeed, till the appearance of Townsend's volumes, in 1863, it is believed that the only work in the English language on this subject was that of Dr. Mulcahy, and in any language that of Chasles, "*Traité de Géométrie Supérieure,*" which had then been published but little more than a decade.

Mathematicians have not only introduced a new language, which, taken in connexion with the new processes,

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makes modern mathematics absolutely unintelligible to one who has for a few years laid aside such studies, but also new functions, whose theory is regarded as a high subject of research. It would be simple pedantry to attempt an illustration of this in terms of the language itself; but we may select a function which is well known, and, ascending briefly the steps by which it has reached its present development, observe something of the spirit of modern mathematical analysis.

Take the theory of Elliptic Functions. Before the middle of the last century, mathematicians began to investigate the solutions of problems depending on the rectification of elliptic arcs.

Undoubtedly the first definite progress in the right direction was the discovery of Euler, which is recorded in sec 7. of *Novi Comm. Petrop.* for 1758-59, and which gives the integral of the differential equation

$$\frac{mdx}{(a+bx+cx^2+dx^3+ex^4)^{\frac{1}{2}}}=\frac{ndy}{(a+by+cy^2+dy^3+ey^4)^{\frac{1}{2}}}.$$

The next step was taken by Lagrange, who published, in the fourth volume (p. 98) of "*Mélanges de Philosophie et de Mathématique de Turin,*" an *à priori* solution of the same general equation which Euler had solved tentatively for special cases.

In 1775, John Landen published in the "Philosophical Transactions" his theorem, showing that any arc of an hyperbola is equal to the difference of two elliptic arcs. The extension of this theorem relating to the general theory of transformation is still the subject of research among mathematicians, among whom especially may be mentioned Richelot (see "*Die Landensche Transformation,*"

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Königsburg, 1868, also in several volumes of *Crelle's Journal.*)

In 1786, Legendre's first paper upon Elliptic Integrals was presented to the French Academy; and from that time onward, for a space of nearly fifty years, till his death, this subject chiefly engaged his attention; and when, in 1825, he presented to the Académie des Sciences the first volume of his "*Traité des Fonctions Elliptiques,*" it was supposed that the resources of the Integral Calculus in this direction were exhausted.

About this time, however, the young Norwegian Abel appeared upon the field; and, by bringing into his analysis the general Theory of Equations, was enabled to show that what had been done was but a small part of what might be expected; and immediately extended the boundaries of knowledge by proving his theorem for the comparison of all Transcendental Functions whatever, whose differentials are irrational from involving the second root of a rational function of the variable x. This is not the place to describe Abel's theorem; but the great research bestowed by modern mathematicians upon the Abelian Functions serves to show the spirit and line of a particular analysis, and the interest which attaches to a subject, which, under continual expansion for more than a century by minds of the highest mathematical power, still suggests for itself a much greater amplitude.

In the complete works of Abel, by Holmboe, we see the ease and power of that remarkable genius, for whom the principal mathematicians of his age, Poisson, Cauchy, and Legendre, foresaw the wreath of an enduring fame. Of the labors of Jacobi in this direction, whose work,

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"*Nova Fundamenta,*" appeared in the same year of Abel's death, 1829, it is not my intention to speak. Had Abel reached the patriarchal age of Legendre, he would still be living to write theorems for *future* generations. Abel died before he had completed his twenty-seventh year.

In *cahier* 23 of the "*Journal de l'Ecole Polytechnique,*" and in the 9th volume of *Liouville,* and in the 18th and 19th of *Comptes Rendus,* we find Abel's work proved and elucidated by Hermite and Liouville. In these journals, and in *Comptes Rendus* since 1843, the contributions of MM. Serret and Chasles would need especial study. So also "*Theorie der Abelschen Functionen,*" by Clebsch and Gordon, Professors in the University of Giessen (1866), and "*Théorie des Fonctions doublement périodiques et des Fonctions elliptiques,*" by Briot and Bouquet (1859).

It is not necessary to mention the greater number of distinguished Continental writers upon Abelian functions. Neumann of Halle, and Riemann of Tübingen (1863-4), Ivory, Bronwin, and Cayley, of Cambridge, are some of the well known writers upon these functions.

The student, however, should not fail to study the papers of Konigsberger (*Crelle,* Vol. 64) and Weirstrass on the solution of Hyperelliptic Functions (*Crelle,* Vol. 47); nor should a paper by Rosenhain, in "*Mémoires de l'Institut par divers Savans,*" be omitted, as also a report by Russell on Elliptic and Hyperelliptic Integrals before the British Association, from 1870 and now in progress.

But what is the use of such studies? If the array of illustrious names herein given do not sufficiently guarantee their importance, let me say that it is by such abstract and difficult labors men become mathematicians. What

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then? Well, suppose that it is shown that the secular inequalities resulting from the action of one planet on another are the same as if the mass of the disturbing planet were diffused along its orbit in the form of an elliptic ring of variable but indefinitely small thickness, and that it is inquired, what is the attraction exerted by such a ring upon an external point? The problem involves eventually two elliptic integrals, as Gauss shows, of the first and second kinds.

The final application, then, of the higher analysis must be the sufficient answer to all *cui bono* inquirers. Take, for instance, the original Bessel's functions in $L^m_{(z)}$, $Y^m_{(z)}$, and

$$Jn\,(z) = \frac{1}{\pi} \int_0^\pi \cos\,(z \sin \omega - n\omega)\, d\omega,$$

hitherto mostly in the hands of German mathematicians, and successfully applied to the solution of physical problems in heat, electricity, and the investigation of aerial vibrations in cylindrical spaces. A good example and illustration of this function may be seen in "*Studien über Bessell'schen Funktionen,*" by Dr. Eugen Lommel, a paper in *Crelle*, Vol. 56, and one of high value by Strutt of Cambridge.

If the utility, then, of advanced modern mathematical study is not to be doubted, what provision can be made for its wider diffusion?

Now, the work of reducing the higher mathematics to the comprehension of ordinary readers, while confessedly a difficult and generally a thankless undertaking, has in some cases been attended with unlooked-for success.

Bowditch's notes upon "*Mécanique Céleste,*" side by side with his translation; Mrs. Somerville's paraphrase of the

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same original work, and the excessive elementary labors of the Jesuit Fathers upon the *Principia*, were and are rewarded with the strongest expressions of appreciation. And there can be no doubt that similar labors will, in some circles, always be regarded with favor. Students must early know the goal, else their ambition may come too late. The equipment of a mathematican is now a very different thing from what it was thirty, or even ten, years ago. There should be some way by which, in very early years, the broad field of modern mathematics could be entered. Determinants should be taught constantly with common Algebra; Quaternions with Geometry; Trilinear Coordinates with the Cartesian; and Invariants, Co-variants, and Contravariants with the general Theory of Equations.

One grand principle should never be forgotten: the educational value of a subject is greatly modified by the the hands which administer it.

This is conspicuously true in mathematical teaching, whether by books or lectures. Let this be suggested. Every high subject has its easy elementary side, and there it may be pierced. The works of Cremona, Helmholtz, Tait, Sylvester, Clifford, and Cayley, may, in some of their elementary forms, be commingled with ordinary mathematical studies; and thus the ancient tasks of the student will be expanded and enlivened by fresh contributions from the great teachers of the world. Inspiration is needed for study, and study must deepen the inspiration.

The fundamental equations of Quaternions in i, j, k are easily exhibited to a class in Geometry in such manner as to become a source of real pleasure to them; and thus

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they may be incited to learn the power of an instrument which bids fair to stand unrivalled in the field of mathematical physics.

The rich stores of research and discovery entombed in the volumes of the learned societies of Europe, and in the mathematical journals, are something enormous; and my object is to bring, in a more elementary form, some of the more important subjects into a wider notice.

In regard to this tract on Determinants, it is very elementary, and intended to be more suggestive than exhaustive.

The works consulted in its preparation embrace the entire literature of the subject; viz.—The theory and practice of Determinants, by Baltzer, Brioschi, Spottiswoode, Salmon, Trudi, Dodgson (the two latter hardly worth consulting); the numberless papers in *Crelle* and *Liouville*, and in the Proceedings of the Royal Societies, from 1841; also a short account of Functional Determinants in the Analytical Mechanics of Prof. Peirce, of Harvard; and the chapter devoted to the subject by Todhunter, in his Theory of Equations, and those of Boole, Ferrers, and Whitworth.

W. J. W.

15, Regent Square,
London, W.C.; 1875.

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ELEMENTARY DETERMINANTS.

CHAPTER I.

PRINCIPLES.

1. *Definitions.*—The common and general expression for a determinant of the nth order consists of the arrangement of n^2 quantities in n rows and n columns, as follows :—

a_{21} ?

$$\begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{12} & a_{22} & \ldots\ldots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}$$

or, more briefly, $\Sigma\,(\pm\, a_{11}\ a_{22}\ \ldots\ldots\ a_{nn})$,

which is to be understood as expressing the sum of the $1 . 2 . 3 \ldots n$ products obtained by fully permuting the n suffixes, so that each product shall include all the suffixes, and the several products differ from each other by at least one variation of these suffixes.

Every variation in the suffixes introduces a change of sign.

The letters of the expression

$$\Sigma\,(\pm\, a_{11}\ a_{22}\ \ldots\ldots\ a_{nn}),$$

taken from the diagonal of the square, are called the *leading* letters; and these, together with all the others of the determinant, are called the *constituents.*

The products themselves, when formed, are called the *elements.**

* Called by Laplace resultants (*Hist. de l'Acad.* 1772).

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Constituents are called conjugate to each other, when, considered in reference to their respective rows and columns, they hold the same positions.

Reserving other definitions till we have made some development of the subject, let us seek some simple illustrations in the formation of determinants.

2. Let us assume a determinant of two places, or the second order, as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

Writing together the leading letters $a_1 b_2$, we have the first element, or product, and permuting the suffixes we obtain the second $a_2 b_1$, since, by definition, a variation of the suffixes gives a change of sign.* These two products taken together form the *expansion* of the determinant. Hence we write

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1.$$

3. Connecting now these constituents with variables, let us find the conditions of co-existence of the homogeneous equations of the first degree

$$a_1 x + b_1 y = a_2 x + b_2 y,$$

and suppose c to be their common value, then

$$a_1 x + b_1 y = c,$$

$$a_2 x + b_2 y = c.$$

Eliminating y and x, we have

$$\left.\begin{aligned} (a_1 b_2 - a_2 b_1)\, x &= b_2 c - b_1 c \\ \text{and} \qquad (a_1 b_2 - a_2 b_1)\, y &= a_1 c - a_2 c \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

Observe (1) that the coefficient $a_1 b_2 - a_2 b_1$ is common to both

* Laplace has not only stated the rule for the change of signs by *disarrangement*, which he refers to M. Cramer, but proves the more simple rule of Bezout by permutation of the suffixes. (*Hist. de l'Acad.* 1772, p. 295.)

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variables; (2) that this coefficient is identical with the value of the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$,

as given above; it may therefore be written in that form. The same remark will evidently apply to the second members b_2c-b_1c and a_1c-a_2c, and we may write (1) as

$$\left.\begin{aligned} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x &= \begin{vmatrix} c & b_1 \\ c & b_2 \end{vmatrix} \\ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y &= \begin{vmatrix} c & a_2 \\ c & a_1 \end{vmatrix} \end{aligned}\right\} \qquad \text{(2)}.$$

If now we regard $c=0$, the second members of (1) and (2) vanish, and we have simply

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

which must be interpreted as the condition of the co-existence of the two equations, when their second members vanish.

4. It will be sufficiently evident, from what has preceded, how a determinant of the second order, as last written, is to be expanded; viz., by multiplying together the letters of each diagonal, beginning with the upper left-hand corner, and connecting the products with the negative sign. Observing this rule for forming the products, it plainly can make no difference with the result, if we write the rows as columns; as,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

5. It is also evident that the sign of the determinant will change when the rows, or columns, are interchanged; as,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}.^*$$

* Laplace "*Sur le calcul intégral.*"

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6. Let us now examine a determinant of the third order,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1 \begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Each of these three determinants is formed by omitting in succession the row and column which contain a_1, b_1, c_1; *i.e.*, by writing in the above order the remainders of the second and third columns, the third and first, and first and second. The further expansion may be written out by the rule already given for the determinant of the second degree, that is, by diagonal multiplication. Otherwise, we may write down the leading letters a_1, b_2, c_3, and fully permute the suffixes. The sum of all the products thus obtained, with their appropriate signs, will express the true expansion. The rule for the signs being, as has been stated, that every variation of the suffixes yields a change of sign, or that an even number of permutations gives a *plus*, and an odd number, a *minus* sign. The permutation of the suffixes of the diagonal letters ($a_1\, b_2\, c_3$) gives six products, which result corresponds with products obtained from reducing each of the three equivalent determinants as written above; as,

$$a_1 b_2 c_3 - a_1 b_3 c_2 + b_1 c_2 a_3 - b_1 c_3 a_2 + c_1 a_2 b_3 - c_1 a_3 b_2.$$

7. It is easily shown that, if two rows, or two columns, become identical, the determinant vanishes; as,

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

which is perhaps sufficiently obvious without multiplying out at length.

8. We shall now proceed to show how this determinant arises. Having shown how a determinant of the third order may be reduced, the determinant itself being given, let us

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seek, inversely, to construct a function, or its equivalent functions, by an actual process of elimination, such that its several products shall be identical with those of the given determinant.*

Let us seek, for example, the condition of the co-existence of the equations

$$a_1x+b_1y+c_1z = a_2x+b_2y+c_2z = a_3x+b_3y+c_3z.$$

If the common value be zero, then

$$\left.\begin{aligned} a_1x+b_1y+c_1z &= 0 \\ a_2x+b_2y+c_2z &= 0 \\ a_3x+b_3y+c_3z &= 0 \end{aligned}\right\} \cdots\cdots\cdots\cdots\cdots\cdots (3).$$

As to the manner of solving these equations so as to exhibit the required condition, two methods, at least, are open to us.†

(*a*.) We may multiply the second of the equations by l, the third by m, and add; then, whatever the value of the variables, l and m may be so taken as to cause two of their coefficients with which they are multiplied to disappear—that is, two of the coefficients of the second and third equations; and, since the equations are simultaneous, that of the third must vanish also. The equations will now contain only two unknowns, l and m, whence these may be determined from the second and third, and their values substituted will give the desired condition. So far as this relates to elimination, it is similar to the method employed by Laplace, referred to in a note below.

(*b*.) Otherwise, by eliminating alternately y and z from (3),

$$(a_2b_3-a_3b_2)x+(c_2b_3-b_2c_3)z = 0,$$

$$(a_2c_3-a_3c_2)x+(b_2c_3-b_3c_2)y = 0,$$

* This was also exhibited by Laplace, and its application to the resolution of linear equations. It might be of interest to compare the method of Lagrange, in his Memoir on the "Movement of the nodes and inclination of the orbits of planets," with the theorem of Malmsten for finding particular integrals by determinants.

† See Ferrers, Salmon, and Tait, on Determinants.

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which, remembering to change signs in transposing, may evidently be written

$$\frac{x}{b_2c_3-c_2b_3}=\frac{y}{a_3c_2-a_2c_3}=\frac{z}{a_2b_3-a_3b_2}.$$

Dividing now the terms of the first of (3) respectively by these equals, we have

$$a_1(b_2c_3-c_2b_3)+b_1(a_3c_2-a_2c_3)+c_1(a_2b_3-a_3b_2)=0 \;\ldots\ldots\; (4).$$

Had we divided in the same manner the terms of the second and third equations of (3), we should have found identical relations; and, since the equations are simultaneous, we have therefore found the required condition. If now we perform in (4) the multiplications indicated, we shall have six products identical with those obtained above.

We see also that (4) may be written (Art. 5)

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}+b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}+c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}=0,$$

and hence also
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

9. If the products are written out, by permuting the suffixes, it is only necessary to observe the cyclic order; thus, if we have $a_1(b_2c_3)$ the next function of the order or line a must be $a_2(b_3c_1)$, and the third $a_3(b_1c_2)$.

Also that, while $(a_1b_2c_3)$ is of course identical with itself, it indicates the determinant equally in each of two positions; *i.e.*,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

$$\text{The one } = (a_1b_2c_3)=a_1(b_2c_3)+b_1(c_2a_3)+c_1(a_2b_3),$$

$$\text{the other } = (a_1b_2c_3)=a_1(b_2c_3)+a_2(b_3c_1)+a_3(b_1c_2).$$

Taken together the products are equal, but the corresponding

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terms, after the first, are dissimilar. The reason of this remark will be obvious, when it is considered that the products may be derived otherwise than by permuting the suffixes.

10. We are in a position now to illustrate one or two important uses of determinants as a system of notation. The equation, for instance, of the straight line passing through two given points, may be written as a determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & y_2 \\ x & x_1 & x_2 \end{vmatrix} = 0.$$

In this form it is easily remembered; while, for practical use, greater brevity and clearness will be ensured. Suppose one of the points, as $(x_2 y_2)$, is changed to the origin; then, since its coordinates must vanish, the determinant becomes

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & 0 \\ x & x_1 & 0 \end{vmatrix} = 0.$$

Now, we know the equation of a line passing through the origin and a given point to be $y = \frac{y_1}{x_1} x$, which must be the value of the determinant if our notation holds true. Hence we write

$$\begin{vmatrix} 1 & 1 & 1 \\ y & y_1 & 0 \\ x & x_1 & 0 \end{vmatrix} = \begin{vmatrix} y & y_1 \\ x & x_1 \end{vmatrix} = 0;$$

and we may thus here state what will be found true generally, *that if all the constituents but one of a column or row become zero, both the column and row which contain that constituent may be erased from the determinant.*

A second illustration may be found in the expression for the area of a triangle in terms of the coordinates of its vertices, the axes being rectangular.* This may be written as the

* Salmon's Conics, p. 30.

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foregoing, with an additional suffix, as,

$$\frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ y_1 & y_2 & y_3 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0.$$

If, now, two of the points, as $(x_1\, y_1)$, $(x_2\, y_2)$, be connected with the origin, the coordinates evidently of the other vertex become zero, and we have, therefore, simply

$$\frac{1}{2}\begin{vmatrix} y_1 & y_2 \\ x_1 & x_2 \end{vmatrix} = 0.$$

Other illustrations will occur to the reader.

11. *If any row or column be multiplied by any quantity, the determinant is multiplied by that quantity.*

$$x\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} xa & b & c \\ xa_1 & b_1 & c_1 \\ xa_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} ax & bx & cx \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \;\ldots\ldots(1).$$

A negative sign placed before the determinant is equivalent to interchanging one row or column with another parallel to it.

$$-\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ a & b \end{vmatrix} = \begin{vmatrix} b & a \\ b_1 & a_1 \end{vmatrix} \;\ldots\ldots\ldots\ldots\; (2).$$

It follows, from (1), that

$$\begin{vmatrix} x^2 & xy & xz \\ x & y_1 & z_1 \\ x & y_2 & z_2 \end{vmatrix} = x^2\begin{vmatrix} 1 & y & z \\ 1 & y_1 & z_1 \\ 1 & y_2 & z_2 \end{vmatrix};$$

and, from (1) and (2), that

$$\begin{vmatrix} a & b & c \\ a & b & c_1 \\ a & b & c_2 \end{vmatrix} = ab\begin{vmatrix} 1 & c & 1 \\ 1 & c_1 & 1 \\ 1 & c_2 & 1 \end{vmatrix}.$$

All these results are so nearly self-evident, or so easily verified, that it is only necessary to write them down.

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12. *Minors.* The determinant

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = a\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + b\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} + c\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \ldots (1)$$

may be written more briefly as

$$\Delta = aA + bB + cC,$$

where Δ represents the primitive determinant, and A, B, and C the several minors formed, as is evident, by omitting in turn the column and row which contain a, b, c. The determinant may also be written $\Delta = aA + a_1B_1 + a_2C_1$, where A, B_1, and C_1 represent the minors when the rows of the determinant are written as columns—

$$\begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} = a\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} + a_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} + a_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} \ldots (2).$$

In comparing (1) and (2), it will be seen that the first minor in each of the two sets of minors is the same, but the others are unlike. It is important to observe this difference. The practical use will be seen in the solution of the following equations :

$$\begin{aligned} ax + by + cz &= e, \\ a_1x + b_1y + c_1z &= e_1, \\ a_2x + b_2y + c_2z &= e_2. \end{aligned}$$

Multiply the first by A, the second by B_1, and the third by C_1, and add, and we have $\Delta x = Ae + B_1e_1 + C_1e_2$, since y and z vanish. In like manner the values of y and z may be found.

It might be necessary to some readers to see the entire process written out; thus,

$$\begin{aligned} aAx + bAy + cAz &= e\,A, \\ a_1B_1x + b_1B_1y + c_1B_1z &= e_1B_1, \\ a_2C_1x + b_2C_1y + c_2C_1z &= e_2C_1\,; \end{aligned}$$

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or $$a\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} x + b\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} y + c\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} z = e\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$a_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} x + b_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} y + c_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix} z = e_1\begin{vmatrix} b_2 & b \\ c_2 & c \end{vmatrix}$$

$$a_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} x + b_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} y + c_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix} z = e_2\begin{vmatrix} b & b_1 \\ c & c_1 \end{vmatrix}$$

Adding and combining, we have

$$\begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} x + \begin{vmatrix} b & b_1 & b_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} y + \begin{vmatrix} c & c_1 & c_2 \\ c & c_1 & c_2 \\ b & b_1 & b_2 \end{vmatrix} z = \begin{vmatrix} e & e_1 & e_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix}.$$

The coefficients of y and z having two parallel lines identical vanish, and we have, changing the rows to columns for the final expression,

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} x = \begin{vmatrix} e & b & c \\ e_1 & b_1 & c_1 \\ e_2 & b_2 & c_2 \end{vmatrix}$$

13. *If the constituents of any determinant be resolvable into the sum of n other constituents, the determinant is resolvable into the sum of n other determinants.*

Let $\Delta = aA + bB + cC$, where A, B, C have the meaning of Art 12. Increase a, b, c by x, y, z, respectively,*

$$\Delta_1 = (a+x)A + (b+y)B + (c+z)\,C$$

$$= (a+x)\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} + (b+y)\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} + (c+z)\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

Hence $$\Delta_1 = \begin{vmatrix} a+x & a_1 & a_2 \\ b+y & b_1 & b_2 \\ c+z & c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a & a_1 & a_2 \\ b & b_1 & b_2 \\ c & c_1 & c_2 \end{vmatrix} + \begin{vmatrix} x & a_1 & a_2 \\ y & b_1 & b_2 \\ z & c_1 & c_2 \end{vmatrix}$$

If we had used the sign of multiplication, or division,

* Salmon's Deter. p. 10.

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between the quantities a and x, b and y, &c., we should have reached results already pointed out in Art. 11.

14. Since identical parallel lines in a determinant cause it to vanish, we might infer the same result if two given lines differ only by a constant factor; as,

$$\begin{vmatrix} ax & a \\ ax & a \end{vmatrix} = x \begin{vmatrix} a & a \\ a & a \end{vmatrix} = 0.$$

So also, having in mind the proof of the last Art., we might show that, when the sum of several lines differs from the given lines only by constant factors, the same result will follow; as,

$$\begin{vmatrix} la+ma_1 & a & a_1 \\ lb+mb_1 & b & b_1 \\ lc+mc_1 & c & c_1 \end{vmatrix} = l \begin{vmatrix} a & a & a_1 \\ b & b & b_1 \\ c & c & c_1 \end{vmatrix} + m \begin{vmatrix} a_1 & a & a_1 \\ b_1 & b & b_1 \\ c_1 & c & c_1 \end{vmatrix}$$

In the same manner, if to any line we add the sum of the other lines separately, or increased by constant factors, the determinant will vanish;* thus,

$$\begin{vmatrix} 1+na+mb & a & b \\ 1+na_1+mb_1 & a_1 & b_1 \\ 1+na_2+mb_2 & a_2 & b_2 \end{vmatrix} = \begin{vmatrix} 1 & a & b \\ 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \end{vmatrix} + \begin{vmatrix} na+mb & a & b \\ na_1+mb_1 & a_1 & b_1 \\ na_2+mb_2 & a_2 & b_2 \end{vmatrix}$$

And, as the last determinant vanishes, the remaining one is evidently the original. Hence we may easily verify the following:

$$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a-(b+c) & b & c \\ a_1-(b_1+c_1) & b_1 & c_1 \\ a_2-(b_2+c_2) & b_2 & c_2 \end{vmatrix} = \begin{vmatrix} a-a_2 & b-b_2 & c-c_2 \\ a_1-a_2 & b_1-b_2 & c_1-c_2 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

&c. &c. &c.

15. *Determinants of the fourth order.*

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

expresses the condition of the co-existence of four homogeneous

* Spottiswoode's Deter. p. 18.

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equations of the first degree when the second members vanish (Art. 8). We may regard it as the sum of four determinants of the third order, each of which gives three other partial determinants, and each of these in turn gives two products. The whole number of products of a determinant of this order will therefore be

$$1 . 2 . 3 . 4,$$

a result identical with the number obtained by permuting the suffixes, as $a_1b_2c_3d_4$. This, as a determinant, may be expressed as four partial determinants,

$$a_1 (b_2 c_3 d_4), \quad a_2 (b_3 c_4 d_1), \quad a_3 (b_4 c_1 d_2), \quad a_4 (b_1 c_2 d_3).$$

This result may be obtained by the actual solution of four equations with four variables, or the law of formation as seen in the case of three unknowns would enable us to write

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & . & . \\ . & c_3 & . \\ . & . & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_3 & . & . \\ . & c_4 & . \\ . & . & d_1 \end{vmatrix} + a_3 \begin{vmatrix} b_4 & . & . \\ . & c_1 & . \\ . & . & d_2 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & . & . \\ . & c_2 & . \\ . & . & d_3 \end{vmatrix}$$

with this exception, we could not tell what signs to write before them.

These must be determined by considering the number of permutations which arise between $a_1 (b_2 c_3 d_4)$ and $a_2 (b_3 c_4 d_1)$. First, considering $a_1 (b_2 c_3 d_4)$, which we assume to be *plus*, we exchange suffixes with a and b, which gives the required suffix for a. Then, let b and d exchange, which gives d_1, finally b and c, when the required element is reached in all three permutations; and therefore the sign must be negative, by definition, since the number is odd. The third element is obtained from $a_1 (b_2 c_3 d_4)$ by permuting the suffixes of a and c, and b and d, an even number; and therefore the sign to be prefixed is *plus*.

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The fourth element with three permutations is found, in the same manner, to be negative; and thus the whole number of products of this, and any other determinant, of any order, assuming the law of formation to be general, may be written out at once.

16. Before proceeding further, it may be interesting to work one or two examples for the sake of illustrating the reduction of determinants of the third order as exhibited under Art. 14, and show how the same principles may be applied to those of four places.

Ex. 1.—Let it be required to find the equation of a circle through three points, say (2, 3), (4, 5), (6, 1).

We shall evidently obtain three equations by substituting successively these coordinates of the three points in the general equation $x^2+y^2+2ax+2by+c=0$, viz.,

$$\begin{aligned} 4a+\ \ 6b+c &= -13, \\ 8a+10b+c &= -41, \\ 12a+\ \ 2b+c &= -37. \end{aligned}$$

To obtain a, b, c we have

$$\begin{vmatrix} 4 & 6 & 1 \\ 8 & 10 & 1 \\ 12 & 2 & 1 \end{vmatrix} a = \begin{vmatrix} -13 & 6 & 1 \\ -41 & 10 & 1 \\ -37 & 2 & 1 \end{vmatrix}$$

$$8\begin{vmatrix} -2 & 2 & 0 \\ -1 & 4 & 0 \\ 3 & 1 & 1 \end{vmatrix} a = 2\begin{vmatrix} -13 & 3 & 1 \\ -28 & 2 & 0 \\ -24 & -2 & 0 \end{vmatrix}$$

$$8\begin{vmatrix} -2 & 2 \\ -1 & 4 \end{vmatrix} a = 8\begin{vmatrix} -7 & 2 \\ -6 & -2 \end{vmatrix}$$

$$a = -\frac{13}{3}.$$

Explanation.—The Δ, coefficient of a, has the two factors 2 and 4. The bottom row subtracted from each of the other rows gives zero for a constituent in two places, which, by Art. 8, causes Δ to reduce to the 2nd or lowest order.

The absolute term of the equation is first factored, and then the upper row is taken from each of the others, when it, like the other, reduces to a Δ of 2nd degree.

For the other values of the unknowns, we write for the determinant, which has been found to be -48, successively,

$$\Delta b = \begin{vmatrix} -13 & 1 & 4 \\ -41 & 1 & 8 \\ -37 & 1 & 12 \end{vmatrix} \text{ and } \Delta c = \begin{vmatrix} -13 & 4 & 6 \\ -41 & 8 & 10 \\ -37 & 12 & 2 \end{vmatrix}$$

$$b = -\frac{8}{3}, \qquad c = \frac{61}{3}.$$

From these values the required equation can be formed. The reductions of the right members of these equations are effected in the same manner as for the value of a, almost by simple inspection.

Ex. 2.—Take the quadric (given by Dr. Salmon, p. 168 of his *Solid Geometry*)

$$7x^2+6y^2+5z^2-4yz-4xy+10x+4y+6z+4=0,$$

differentiate it with respect to its variables, and we shall have

$$7x-2y+5=0,$$
$$-2x+6y-2z+2=0,$$
$$-2y+5z+3=0,$$
$$5x+2y+3z+4=0.$$

The determinant will then be, when written at length,

$$\begin{vmatrix} 7 & -2 & 0 & 5 \\ -2 & 6 & -2 & 2 \\ 0 & -2 & 5 & 3 \\ 5 & 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & -2 & 0 & 5 \\ -4 & 6 & -2 & 2 \\ -3 & -2 & 5 & 3 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -6 & -6 & -3 \\ 0 & 14 & 10 & 18 \\ 0 & 4 & 14 & 15 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

Explanation.—Obtained by taking the last column from the first; next, twice the bottom row from the first, and adding four and three times the same to the second and third rows; then thrice the last column from the first and second.

It is to be especially observed that the sign changes in the determinant when the factor 12 appears. The reason is obvious, since the whole determinant is

$$a_1(b_2c_3d_4)-a_2(b_3c_4d_1)+a_3(b_4c_1d_2) -a_4(b_1c_2d_3),$$

and $\qquad a_1=a_2=a_3=0,$

while $a_4=1$, and therefore in this case the determinant reduces to

$$-(b_1c_2d_3).$$

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$$= 12 \begin{vmatrix} 3 & 3 & 1 \\ 7 & 5 & 6 \\ 2 & 7 & 5 \end{vmatrix} = 12 \begin{vmatrix} 0 & 0 & 1 \\ -11 & -13 & 6 \\ -13 & -8 & 5 \end{vmatrix}$$

$$= 12 \begin{vmatrix} -11 & -13 \\ -13 & -8 \end{vmatrix} = -972.$$

Ex. 3.—To find an expression for three points in involution.

Substitute, in the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = 0,$$

$$x_1 = a_1 + a_2, \quad x_2 = b_1 + b_2, \quad x_3 = c_1 + c_2.$$

$$y_1 = a_1 a_2, \quad y_2 = b_1 b_2, \quad y_3 = c_1 c_2.$$

and we have

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1+a_2 & b_1+b_2 & c_1+c_2 \\ a_1 a_2 & b_1 b_2 & c_1 c_2 \end{vmatrix} = (c_1-a_2)(b_1-c_2)(a_1-b_2) + (c_2-a_1)(b_2-c_1)(a_2-b_1).$$

Ex. 4.—The following solution is given of the determinant proposed by Dr. Salmon (p. 12, *Deter.*)

$$\begin{vmatrix} 25 & -15 & 23 & -5 \\ -15 & -10 & 19 & 5 \\ 23 & 19 & -15 & 9 \\ -5 & 5 & 9 & -5 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 30 & 104 & -50 \\ 0 & -20 & 1 & 15 \\ 0 & 14 & -24 & 14 \\ -5 & 5 & 9 & -5 \end{vmatrix}$$

Explanation.—The sum of 2nd and 4th columns is taken from the 1st, 9 times last row is added to the first, once and twice the last are taken from 2nd and 3rd, the sum of 2nd and 3rd columns is taken from the first, the last row is added to the 1st and 2nd, and finally twice the 2nd row is added to the last. Sign changes twice — 1st, when $-a_4(b_1 c_2 d_3)$ alone remains of the determinant; and, 2nd, when by a still further reduction the determinant becomes $-a_2(a_3 b_1)$.

$$= +5 \begin{vmatrix} 0 & 80 & -36 \\ -12 & -23 & 29 \\ 0 & -70 & 72 \end{vmatrix} = 36 \times -600 \begin{vmatrix} 8 & -1 \\ -7 & 2 \end{vmatrix}$$

$$= -194400.$$

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Ex. 5.—*Notation.*

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

This determinant expresses what will be at once recognised as expressing the elimination of x, y, z from

$$\begin{aligned} lx + my + nz &= 0, \\ l_1 x + m_1 y + n_1 z &= 0, \\ l_2 x + m_2 y + n_2 z &= 0, \end{aligned}$$

or the condition that three straight lines may be parallel to one plane.

Ex. 6.—Let S_1, S_2, S_3 be three circles of the form

$$S_1 = (x - a_1)^2 + (y - b_1)^2 - c_1^2 = 0\,;$$

find the circle orthotomic to these three.

Let S be the required circle; and, since S and S_1 are to be orthotomic, we must have

$$(a_1 - a)^2 + (b_1 - b)^2 = c_1^2 + c^2\,;$$

and, by eliminating a, b, and $(a^2 + b^2 - c^2)$ from the four resulting equations, we have the determinant

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ a_1^2 + b_1^2 - c_1^2 & a_1 & b_1 & 1 \\ a_2^2 + b_2^2 - c_2^2 & a_2 & b_2 & 1 \\ a_3^2 + b_3^2 - c_3^2 & a_3 & b_3 & 1 \end{vmatrix} = 0.$$

17. *Multiplication of Determinants.*—We have now to determine the product of one determinant by another. This we may accomplish by the method of *transformation.**

* Salmon, Spottiswoode, and Tait.

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Let us take two systems of linear equations,

$$\begin{aligned} ax\ +by\ +cz\ &= v, \\ a_1x+b_1y+c_1z &= v_1, \\ a_2x+b_2y+c_2z &= v_2, \end{aligned}$$

and

$$\begin{aligned} dv\ +ev_1\ +fv_2\ &= 0, \\ d_1v+e_1v_1+f_1v_2 &= 0, \\ d_2v+e_2v_1+f_2v_2 &= 0. \end{aligned}$$

Substitute the values of v, v_1, &c. in the second, and, collecting terms, we shall have

$$\begin{aligned} (ad\ +ea_1\ +fa_2\ )\,x+\&c. &= 0, \\ (ad_1+e_1a_1+f_1a_2)\,x+\&c. &= 0, \\ (ad_2+e_2a_1+f_2a_2)\,x+\&c. &= 0. \end{aligned}$$

The condition of coexistence of these equations (Art. 8) will be the determinant

$$\begin{vmatrix} ad\ +a_1e\ +a_2f & bd\ +b_1e\ +b_2f & cd\ +c_1e\ +c_2f \\ ad_1+a_1e_1+a_2f_1 & bd_1+b_1e_1+b_2f_1 & cd_1+c_1e_1+c_2f_1 \\ ad_2+a_1e_2+a_2f_2 & bd_2+b_1e_2+b_2f_2 & cd_2+c_1e_2+c_2f_2 \end{vmatrix} = 0 \ldots (1).$$

But it is evident that these two systems of equations may be treated as one, since the variables they contain are common to both; and we may inquire the condition of coexistence of these six equations

$$\begin{aligned} ax\ +by\ +cz\ -v\qquad\qquad &= 0, \\ a_1x+b_1y+c_1z\qquad -v_1\qquad &= 0, \\ a_2x+b_2y+c_2z\qquad\qquad -v_2 &= 0, \\ dv\ +ev_1\ +fv_2 &= 0, \\ d_1v+e_1v_1+f_1v_2 &= 0, \\ d_2v+e_2v_1+f_2v_2 &= 0. \end{aligned}$$

Here we may say, as before, the condition of co-existence of

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these equations is expressed by the determinant

$$\begin{vmatrix} a & b & c & -1 & 0 & 0 \\ a_1 & b_1 & c_1 & 0 & -1 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & -1 \\ 0 & 0 & 0 & d & e & f \\ 0 & 0 & 0 & d_1 & e_1 & f_1 \\ 0 & 0 & 0 & d_2 & e_2 & f_2 \end{vmatrix} = 0.$$

This determinant may evidently be written

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = A \times D,$$

or $$\begin{vmatrix} a & b & c \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} \times \begin{vmatrix} d & e & f \\ d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \end{vmatrix} \dots\dots\dots\dots\dots\dots (2);$$

but this is no other than the condition expressed by (1); and therefore we say that (1) and (2) must be equal.

That is, the product of one determinant by another *is a determinant whose constituents consist of the sums of the products obtained by multiplying each column of the one determinant by the rows of the other.*

Ex. 1. $$\begin{vmatrix} \cos a & \cos b & \cos c \\ \cos a_1 & \cos b_1 & \cos c_1 \\ \cos a_2 & \cos b_2 & \cos c_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Ex. 2.—The equations of four planes intersecting in a point

are $$\begin{aligned} lx + my + nz + d &= 0, \\ l_1 x + m_1 y + n_1 z + d_1 &= 0, \\ l_2 x + m_2 y + n_2 z + d_2 &= 0, \\ l_3 x + m_3 y + n_3 z + d_3 &= 0, \end{aligned}$$

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and the determinant formed is evident; but if two of the planes pass through the axis of z, we shall have

$$\begin{vmatrix} l & m & n & d \\ l_1 & m_1 & n_1 & d_1 \\ l_2 & m_2 & 0 & 0 \\ l_3 & m_3 & 0 & 0 \end{vmatrix} = 0,$$

which is simply the product of the two determinants

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} \quad \begin{vmatrix} n & d \\ n_1 & d_1 \end{vmatrix} = 0,$$

which may be multiplied by the rule.

Suppose, however, we wished to interpret the latter equation geometrically, in which case we see that either

$$\begin{vmatrix} l_2 & m_2 \\ l_3 & m_3 \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} n & d \\ n_1 & d_1 \end{vmatrix} = 0.$$

The first supposition marks the coincidence of the third and fourth planes; the second that the four planes intersect somewhere in the axis of z.

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CHAPTER II.

FORMS OF INVERSE AND SKEW DETERMINANTS.

18. *Minors as constituents and as differential coefficients.*

We have already seen that a determinant may be written briefly by the aid of its minors as

$$\Delta = aA + bB + cC.$$

But since in any determinant we can interchange parallel lines and obtain the same result with a change of sign, when the number of such interchanges is odd, we can write a determinant of the third order, as above,

$$\Delta = a_1 A_1 + b_1 B_1 + c_1 C_1,$$

as the result of interchanging the first and second rows, and

$$\Delta = a_2 A_2 + b_2 B_2 + c_2 C_2$$

for the like process between the first and third, or evidently, in general,

$$\Delta = a_{1c} A_{1c} + a_{2c} A_{2c} \text{ ...... } a_{nc} A_{nc} \text{ ....(1)},$$

where c is 1 ...... n.

If now we write

$$\begin{vmatrix} A & B & C \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix}$$

we have what is called the *inverse* or reciprocal of a determinant of three places, that is, a determinant consisting of the *minors* corresponding to the *constituents* of the given determinant.

19. If, now, we differentiate (1), of the last Art., in respect to to a_{1c}, we must have

$$\frac{d\Delta}{da_{1c}} = A_{1c}, \quad \frac{d\Delta}{da_{2c}} = A_{2c}, \text{ \&c.}$$

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That is, if we differentiate a determinant in respect to any constituent, the corresponding minor will be the differential coefficient.*

Hence, for a determinant of the nth order, we may write

$$\Delta = a_{1c}\frac{d\Delta}{da_{1c}} + a_{2c}\frac{d\Delta}{da_{2c}} \ldots\ldots a_n \frac{d\Delta}{da_{nc}} \ldots\ldots\ldots\ldots (1).$$

While this is a more cumbrous notation than that which it replaces, it has its advantages, which will become more apparent; for example, it enables us to distinguish, at once, between those determinants which do and do not, identically, vanish.

Since a determinant is the same in the sum of the products (Art. 14), whether we expand in the order of the rows or columns, we may write

$$\Delta = a_{k1}\frac{d\Delta}{da_{k1}} + a\ {}_2\frac{d\Delta}{da_{k2}} \ldots\ldots a_{kn}\frac{d\Delta}{da_{kn}}.$$

It is equally evident, from what has preceded, that

$$a_{1k}\frac{d\Delta}{da_{1c}} + a_{2k}\frac{d\Delta}{da_{2c}} \ldots\ldots a_{nk}\frac{d\Delta}{da_{nc}} = 0,$$

since in these products we have *in fact* introduced into the given determinant a line parallel and identical with some other line, and therefore the determinant in such form vanishes identically.

This may be explained briefly thus: from what has preceded, it is manifest that, when we take the sum of the products of any line—that is, the sum of the products of all the constituents of that line by their corresponding minors—the determinant subsists; but if the minors do not correspond with their constituents, the determinant vanishes identically; hence, in general,

$$a_{1s}\frac{d\Delta}{da_{1c}} + a_{2s}\frac{d\Delta}{da_{2c}} \ldots\ldots a_{ns}\frac{d\Delta}{a_{nc}} = 0.$$

* The notation followed here is the same as that of Jacobi, Baltzer, Spottiswoode, and Brioschi.

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20. We shall fail, perhaps, of our object unless we descend to special cases.

Let us take a determinant of four places

$$\begin{vmatrix} a_{11} & . & . & a_{14} \\ a_{21} & a_{22} & . & a_{24} \\ . & . & . & . \\ a_{41} & . & . & a_{44} \end{vmatrix}$$

The first minor is obtained by erasing one row and one column, the second minor by erasing two rows and two columns. Let A_{11} = the first minor, and A_{22} the second;

then $$\frac{d\Delta}{da_{11}} = A_{11}, \ \frac{d^2\Delta}{da_{11}\,da_{22}} = A_{22};$$

but $$\frac{d^2\Delta}{da_{11}^2} = 0.$$

So also, when we take the second differential in respect to either the first row or column, the result must be the same, since A_{11} does not contain any one of these constituents.

Hence, in general, we may write

$$\frac{d^2\Delta}{da_{11}\,da_{n1}} = 0 \text{ or } \frac{d^2\Delta}{da_{11}\,da_{1n}} = 0.$$

21. Since an interchange of two lines effects a change of sign, we must indicate a corresponding change in the ensuing differential coefficient.

Thus, while $$\frac{d^2\Delta}{da_{11}\,da_{22}} = A_{22},$$

an exchange of a_{22} with a_{12} or a_{21} gives

$$\frac{d^2\Delta}{da_{21}\,da_{12}} = \frac{d^2\Delta}{da_{12}\,da_{21}} = -\frac{d^2\Delta}{da_{11}\,da_{22}} = A_{22},$$

since in either exchange the second minor is not affected, or in general

$$\frac{d^2\Delta}{da_{cc}\,da_{kk}} = -\frac{d^2\Delta}{da_{ck}\,da_{kc}}.$$

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Evidently no *à priori* proof is needed here; a simple induction, as above, is sufficient: or, in other words, the theorem demands only a clear statement, when its truth is at once obvious.

22. In the case of a symmetrical determinant (Art. 1, def.), when $a_{21} = a_{12}$, we shall find, on differentiating the determinant in reference to any conjugate constituent, that the differential coefficient will be doubled, since the constituent function is supposed to enter twice; as, if $a_{12} = a_{21}$ and $a_{13} = a_{31}$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

and we have $\dfrac{d\Delta}{da_{12}} = 2A_{12}$, and, in general,

$$\frac{da_{ck}}{da_{kc}} = 1 \text{ and } \frac{d\Delta}{da_{ck}} = 2\,A_{ck}.$$

23. In the case, then, of a skew,* as the following, when the terms of the leading diagonal are zero, and the conjugates are of opposite signs, as

$$\begin{vmatrix} 0 & a_{12} & . & . & . & a_{1n} \\ a_{21} & 0 & . & . & . & . \\ a_{31} & . & 0 & . & . & . \\ \cdots & \cdots & \cdots & & & \cdots \\ a_{n1} & a_{n2} & . & . & . & 0 \end{vmatrix}$$

$$a_{12} = -a_{21} \text{ and } a_{1n} = -a_{n1};$$

in which case $$\frac{da_{12}}{da_{21}} = -1;$$

consequently $$\frac{d\Delta}{da_{1n}} = A_{1n} - A_{n1} = 0,$$

when the determinant is of the third and every odd order.

When the determinant is skew and of an even order, we shall have

$$A_{12} = -A_{21} \text{ and } \frac{d\Delta}{da_c} = 2A_{ck}.†$$

* Salmon, p. 30; Crelle, Vol. 51, p. 264.
† Baltzer, p. 13.

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That is, *when the skew symmetric determinant is of an odd degree, it vanishes; but if of an even degree, its differential coefficient in respect to any constituent function is equal to twice its corresponding minor.*

24. Referring again to equation (1), Art. 19, we see that, since $\frac{d\Delta}{da_{1c}} = A_{1c}$ is a determinant of the $n-1$ order, it may, as such, have an expansion similar to that equation. If, for example, the original determinant were of the fourth order, $\frac{d\Delta}{da_{1c}}$ would express a determinant whose outer row and column had been erased, in other words, a determinant of the third order.

Let us take, then, $\frac{d\Delta}{da_{k1}}$ to represent generally a determinant of the $n-1$ order, and suppose

$$\Delta = a_{c1}\frac{d\Delta}{da_{c1}} + a_{c2}\frac{d\Delta}{da_{c2}} \;\ldots\ldots\; a_{cn}\frac{d\Delta}{da_{cn}},$$

to represent a determinant of the n^n order; then, if we differentiate this equation in respect to a_{k1}, the left member will be identical with the proposed expression for the determinant of the $n-1$ order; that is,

$$\frac{d\Delta}{da_{k1}} = a_{c1}\frac{d^2\Delta}{da_{c1}\,da_{k1}} + a_{c2}\frac{d^2\Delta}{da_{c2}\,da_{k1}} \;\ldots\ldots\; a_{cn}\frac{d^2\Delta}{da_{cn}\,da_{k1}}.$$

The same equation, differentiated with respect to a_{k2} and a_{kn}, will yield similar expressions for determinants of the $n-1$ order,

$$\frac{d\Delta}{da_{k2}} = a_{c1}\frac{d^2\Delta}{da_{c1}\,da_{k2}} + a_{c2}\frac{d^2\Delta}{da_{c2}\,da_{k2}} \cdots a_{cn}\frac{d^2\Delta}{da_{cn}\,da_{k2}},$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$\frac{d\Delta}{da_{kn}} = a_{c1}\frac{d^2\Delta}{da_{c1}\,da_{kn}} + \;\ldots\ldots\ldots\ldots\ldots\ldots\; a_{cn}\frac{d^2\Delta}{da_{cn}\,da_{kn}}.$$

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25. Remembering that the determinant subsists when the constituent function, and the function of the differential coefficient, as factors, are identical (Art. 18), we may write

$$\Delta = a_{1c}\frac{d\Delta}{da_{1c}} + a_{2c}\frac{d\Delta}{da_{2c}} \;\ldots\ldots\; a_{nc}\frac{d\Delta}{da_{nc}};$$

but
$$0 = a_{11}\frac{d\Delta}{da_{1c}} + a_{21}\frac{d\Delta}{da_{2c}} \;\ldots\ldots\; a_{n1}\frac{d\Delta}{da_{nc}},$$

when c and 1 are different.

We shall continue to use the differential notation, and apply it to the minors of the reciprocal of a determinant, as

$$\begin{vmatrix} \frac{d\Delta}{da_{11}} & \frac{d\Delta}{da_{12}} & \ldots\ldots & \frac{d\Delta}{da_{1n}} \\ \frac{d\Delta}{da_{21}} & \frac{d\Delta}{da_{22}} & \ldots\ldots & \frac{d\Delta}{da_{2n}} \\ \ldots & \ldots & & \ldots \\ \frac{d\Delta}{da_{n1}} & \frac{d\Delta}{da_{n2}} & \ldots\ldots & \frac{d\Delta}{da_{nn}} \end{vmatrix}.$$

We might use a different notation, as

$$\begin{vmatrix} A_{11} & A_{12} & \ldots\ldots & A_{1n} \\ \ldots & \ldots & & \ldots \\ \ldots & \ldots & & \ldots \\ A_{n1} & A_{n2} & \ldots\ldots & A_{nn} \end{vmatrix}$$

but we prefer to familiarize the reader with the one we have adopted.

26. We now propose the following theorem:—*Any determinant other than skew, multiplied by its second differential coefficient, is equal to the difference of the products of the differential coefficients* a_{11}, a_{12}, a_{21}, a_{22}, *taken as conjugates.*

Confining, for the present, the demonstration to a particular case, let us write

$$0 = a_{11}\frac{d\Delta}{da_{12}} + a_{21}\frac{d\Delta}{da_{22}} \;\ldots\ldots\; a_{41}\frac{d\Delta}{da_{42}},$$

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$$\Delta = a_{12}\frac{d\Delta}{da_{12}} + a_{22}\frac{d\Delta}{da_{22}} \;\ldots\ldots\; a_{42}\frac{d\Delta}{da_{42}},$$

$$\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots$$

$$0 = a_{14}\frac{d\Delta}{da_{12}} + a_{24}\frac{d\Delta}{da_{22}} \;\ldots\ldots\; a_{44}\frac{d\Delta}{da_{42}}.$$

Multiply these equations by

$$\frac{d^2\Delta}{da_{11}\,da_{21}}, \quad \frac{d^2\Delta}{da_{11}\,da_{22}} \;\ldots\ldots\; \frac{d^2\Delta}{da_{11}\,da_{24}},$$

respectively; and, adding the results,

$$\Delta\frac{d^2\Delta}{da_{11}\,da_{22}} = \left\{\begin{array}{l} \left(a_{11}\dfrac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\dfrac{d^2\Delta}{da_{11}\,da_{22}} \;\ldots\ldots\; a_{14}\dfrac{d^2\Delta}{da_{11}\,da_{24}}\right)\dfrac{d\Delta}{da_{12}} \\ \left(a_{21}\dfrac{d\Delta}{da_{11}\,da_{21}} + a_{22}\dfrac{d^2\Delta}{da_{11}\,da_{22}} \;\ldots\ldots\; a_{24}\dfrac{d^2\Delta}{da_{11}\,da_{24}}\right)\dfrac{d\Delta}{da_{22}} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \left(a_{41}\dfrac{d^2\Delta}{da_{11}\,da_{21}} + a_{42}\dfrac{d^2\Delta}{da_{11}\,da_{22}} \;\ldots\ldots\; a_{42}\dfrac{d^2\Delta}{da_{11}\,da_{24}}\right)\dfrac{d\Delta}{da_{42}} \end{array}\right\}.$$

The right member may be reduced as follows:—The first parenthesis becomes, by making one interchange of suffixes,

$$\left(a_{11}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\frac{d^2\Delta}{da_{21}\,da_{12}} \;\ldots\ldots\; a_{14}\frac{d^2\Delta}{da_{14}\,da_{21}}\right).$$

But $$\Delta = a_{11}\frac{d\Delta}{da_{11}} + a_{12}\frac{d\Delta}{da_{12}} \;\ldots\ldots\; a_{14}\frac{d\Delta}{da_{14}},$$

and $$-\frac{d\Delta}{da_{21}} = a_{11}\frac{d^2\Delta}{da_{11}\,da_{21}} + a_{12}\frac{d^2\Delta}{da_{12}\,da_{21}} \;\ldots\ldots\; a_{14}\frac{d^2\Delta}{da_{14}\,da_{21}},$$

therefore $-\dfrac{d\Delta}{da_{21}}$ is the value of the parenthesis.

In the same manner, the second is found equal to $\dfrac{d\Delta}{da_{11}}$; and so also the third and fourth, *without change of sign.* That is, the values of the third and fourth parentheses appear to have the same sign. The *essential* sign must be determined from the rule of signs.

In this case we remember that

$$a_1(b_2 c_3 d_4) - a_2(b_3 c_4 d_1) + a_3(b_4 c_1 d_2) - a_4(b_1 c_2 d_3).$$

The parentheses after the second therefore destroy each other. The multipliers used interpose to change this order in the first and second, and hence we write as the result

$$\Delta \frac{d^2\Delta}{da_{11}\, da_{22}} = \frac{d\Delta}{da_{11}} \cdot \frac{d\Delta}{da_{22}} - \frac{d\Delta}{da_{21}} \cdot \frac{d\Delta}{da_{12}}.$$

This proof might, it is evident, have been made general. It is now, however, in a form to be readily verified.

27. Theorem second.—*A determinant formed from the first differential coefficients of the given determinant may be expressed in terms of the given determinant, and is equal to that determinant involved to a degree one less than its number of places.*

Let
$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{vmatrix}$$

be the given determinant.

Its first differential coefficients, taken in order and arranged in square form, will then be

$$\Delta_1 = \begin{vmatrix} \frac{d\Delta}{da_{11}} & \frac{d\Delta}{da_{12}} & \ldots\ldots & \frac{d\Delta}{da_{1n}} \\ \frac{d\Delta}{da_{21}} & \frac{d\Delta}{da_{22}} & \ldots\ldots & \frac{d\Delta}{da_{2n}} \\ \ldots & \ldots & & \ldots \\ \frac{d\Delta}{da_{n1}} & \frac{d\Delta}{da_{n2}} & \ldots\ldots & \frac{d\Delta}{da_{nn}} \end{vmatrix}$$

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Let now Δ_1 be multiplied by Δ, and we shall have

$$\Delta\Delta_1 =$$

$$\begin{vmatrix} a_{11}\dfrac{d\Delta}{da_{11}}+\ldots a_{1n}\dfrac{d\Delta}{da_{1n}} & a_{21}\dfrac{d\Delta}{da_{11}}+\ldots a_{2n}\dfrac{d\Delta}{da_{1n}} & \ldots\ldots & a_{n1}\dfrac{d\Delta}{da_{11}}+\ldots \&c. \\ a_{11}\dfrac{d\Delta}{da_{21}}+\ldots a_{1n}\dfrac{d\Delta}{da_{2n}} & a_{21}\dfrac{d\Delta}{da_{21}}+\ldots a_{2n}\dfrac{d\Delta}{da_{2n}} & \ldots\ldots & \&c. \quad \&c. \\ \ldots \quad \ldots \quad \ldots & \ldots \quad \ldots \quad \ldots & \ldots & \ldots \quad \ldots \\ a_{11}\dfrac{d\Delta}{da_{n1}}+\ldots a_{1n}\dfrac{d\Delta}{da_{nn}} & \&c. \qquad \&c. & & a_{n1}\dfrac{d\Delta}{da_{n1}}+\ldots \&c. \end{vmatrix}$$

Observing these products, it will be seen that all except those of the leading diagonal vanish identically; and hence we have

$$\Delta\Delta_1 = \begin{vmatrix} \Delta & 0 & \ldots & 0 \\ 0 & \Delta & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \Delta \end{vmatrix} = \Delta^n,$$

or
$$\Delta_1 = \Delta^{n-1},$$

which was to be proved.

28. We shall now begin to introduce, as we proceed with the general theory, some of the geometrical uses of determinants.

Mr. Spottiswoode, in Vol. 51, p. 262, and Prof. Cayley, in 32nd Vol. of *Crelle*, have discussed the subject of orthogonal substitutions in connection with skew determinants.*

We have already given a definition of a skew determinant; we will now show how to effect an orthogonal transformation of the third order, and express the values of the nine direction-cosines in terms of three independent variables, or in general how to connect n^2 quantities by $\frac{1}{2}n\ (n+1)$ relations,

* On the number of linear substitutions, see *Journal de l'Ecole Polytechnique*, Tom. 22, 38 cahier.

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$\frac{1}{2}n(n-1)$ of them only being independent. Let us, for example, write the following linear equations:

$$\begin{aligned}x &= a_{11}u + a_{12}v + a_{13}w,\\ y &= a_{21}u + a_{22}v + a_{23}w,\\ z &= a_{31}u + a_{32}v + a_{33}w,\end{aligned}$$

and a derived system

$$\begin{aligned}X &= a_{11}u + a_{21}v + a_{31}w,\\ Y &= a_{12}u + a_{22}v + a_{32}w,\\ Z &= a_{13}u + a_{23}v + a_{33}w,\end{aligned}$$

where we will suppose $a_{ik} = -a_{ki}$ and $a_{ii} = 1$;

therefore, by addition, we have at once

$$x + X = 2u, \quad y + Y = 2v, \quad z + Z = 2w.$$

If, in the first system, we find the values of u, v, w, which we do by multiplying the equations respectively by

$$\frac{d\Delta}{da_{11}}, \ \frac{d\Delta}{da_{21}}, \ \frac{d\Delta}{da_{31}},$$

and adding, when

$$\Delta u = \frac{d\Delta}{da_{11}}x + \frac{d\Delta}{da_{21}}y + \frac{d\Delta}{da_{31}}z,$$

and, by a similar process, we obtain

$$\Delta v = \frac{d\Delta}{da_{12}}x + \frac{d\Delta}{da_{22}}y + \frac{d\Delta}{da_{32}}z,$$

$$\Delta w = \frac{d\Delta}{da_{13}}x, \ \text{\&c.};$$

whence, by substituting for the values of u, v, w, $u = \frac{x+X}{2}$, &c., we obtain

$$\Delta X = \left(2\frac{d\Delta}{da_{11}} - \Delta\right)x + 2\frac{d\Delta}{da_{21}}y + 2\frac{d\Delta}{da_{31}}z,$$

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$$\Delta Y = 2\frac{d\Delta}{da_{12}}x + \left(2\frac{d\Delta}{da_{22}} - \Delta\right)y + 2\frac{d\Delta}{da_{32}}z,$$

$$\Delta Z = 2\frac{d\Delta}{da_{13}}x + 2\frac{d\Delta}{da_{23}}y + \left(2\frac{d\Delta}{da_{33}} - \Delta\right)z.$$

Treating the second system in the same manner, we find

$$\Delta u = \frac{d\Delta}{da_{11}}X + \&c.,$$

$$\Delta v = \frac{d\Delta}{da_{21}}Y + \&c.,$$

$$\Delta w = \frac{d\Delta}{da_{31}}Z + \&c.;$$

and also, by substitution, taking value of x and instead of X, we find

$$\Delta x = \left(2\frac{d\Delta}{da_{11}} - \Delta\right)X + 2\frac{d\Delta}{da_{12}}Y + 2\frac{d\Delta}{da_{13}}Z,$$

$$\Delta y = 2\frac{d\Delta}{da_{21}}X + \&c.,$$

$$\Delta z = 2\frac{d\Delta}{da_{31}}X + \&c.,$$

or, more symmetrically,

$$\left.\begin{aligned} x &= c_{11}X + c_{12}Y + c_{13}Z \\ y &= c_{21}X + c_{22}Y + c_{23}Z \\ z &= c_{31}X + c_{32}Y + c_{33}Z \end{aligned}\right\} \dots\dots\dots\dots (1),$$

and

$$\left.\begin{aligned} X &= c_{11}x + c_{21}y + c_{31}z \\ Y &= c_{12}x + c_{22}y + c_{32}z \\ Z &= c_{13}x + c_{23}y + c_{33}z \end{aligned}\right\} \dots\dots\dots\dots (2),$$

where

$$\frac{2\dfrac{d\Delta}{da_{11}} - \Delta}{\Delta} = c_{11}, \quad \frac{2\dfrac{d\Delta}{da_{22}} - \Delta}{\Delta} = c_{22}, \text{ and } \frac{2\dfrac{d\Delta}{da_{12}}}{\Delta} = c_{12}, \&c.$$

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Now, if (1) and (2) are connected by an orthogonal substitution, we must have, by Solid Geom.,

$$c_{11}^2+c_{12}^2+c_{13}^2=1,$$
$$c_{11}c_{21}+c_{12}c_{22}+c_{13}c_{23}=0, \text{ \&c. \&c.}$$

That is, the sums of the squares of the direction-cosines = 1, and the sums of their products taken two and two = 0, when the axes are rectangular. But these results immediately follow, if we substitute (2) in (1).

Proceeding now to give to c values corresponding to any given case, we see that the determinant must be analogous to the following *

$$\Delta=\begin{vmatrix} 1 & n & -m \\ -n & 1 & l \\ m & -l & 1 \end{vmatrix}=1+l^2+m^2+n^2,$$

and, forming the minors,

$$\begin{vmatrix} 1+l^2 & lm+n & nl-m \\ lm-n & 1+m^2 & mn+l \\ nl+m & mn-l & 1+n^2 \end{vmatrix}$$

and $$\frac{2\dfrac{d\Delta}{da_{11}}-\Delta}{\Delta}=c_{11}=\frac{1+l^2-m^2-n^2}{1+l^2+m^2+n^2}=c_{22}=c_{33},$$

$$\frac{2\dfrac{d\Delta}{da_{12}}}{\Delta}=c_{12}=\frac{2\,(lm+n)}{1+l^2+m^2+n^2}, \text{ \&c. \&c.*}$$

where c_{11}, c_{12}, &c. represent the values of the nine direction-cosines in the given transformation.

29. Ex. 1. We may find an illustration of what has gone before in the following well-known geometrical relations.

* The values of l, m, n are $a \tan \frac{1}{2}\theta$, $b \tan \frac{1}{2}\theta$, $c \tan \frac{1}{2}\theta$, where the system is revolved through an angle θ, the direction-cosines of the old axes being a, b, c. (*Crelle*, vol. 51, p. 263.)

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Suppose lmn, $l_1m_1n_1$, $l_2m_2n_2$ the direction-cosines of three right lines in reference to their three rectangular axes; a_1, a_2, a_3 the angles included between them:

$$\begin{aligned} l^2+m^2+n^2 &= 1, \quad ll_1+mm_1+nn_1 = \cos a_1,\\ l_1^2+m_1^2+n_1^2 &= 1, \quad ll_2+mm_2+nn_2 = \cos a_2,\\ l_2^2+m_2^2+n_2^2 &= 1, \quad l_1l_2+m_1m_2+n_1n_2 = \cos a_3. \end{aligned}$$

Now we are enabled to write

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & \cos a_3 & \cos a_2 \\ \cos a_3 & 1 & \cos a_1 \\ \cos a_2 & \cos a_1 & 1 \end{vmatrix}.$$

For the above equations are true for every value of a_1, a_2, a_3, and therefore true when a_1 &c. $= 0$, as

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

which conforms to the condition, and is true when the lines are at right angles to each other, giving a determinant which has already been noticed (Art. 16).

Ex. 2. Another illustration is afforded by a determinant which is related to equations of a higher character than we had purposed to introduce at this stage of our progress, but we will just notice it.

Suppose a function of l is expressed in the following determinant,

$$\begin{vmatrix} a-l & d & e \\ d & b-l & f \\ e & f & c-l \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and suppose this function be multiplied by a function of $-l$; we may then write as the result

$$f(-l).f(l) = \begin{vmatrix} A-l^2 & D & E \\ D & B-l^2 & F \\ E & F & C-l^2 \end{vmatrix} \quad \ldots\ldots\ldots(2).$$

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Determinant (1) expresses an equation of frequent occurrence in mathematical physics, as an instance of which the reader may examine Laplace's equation in g on the secular inequalities of the planets (*Mécanique Céleste*, Bk. II. sec. 56.)

Are the roots of such an equation *real?* Special cases had, of course, been resolved by the older mathematicians, as Cauchy and others; but the method by Sylvester (*Philosophical Mag.* 1852), depending upon the rule for the multiplication of determinants, is more simple and elegant. The method is shown above in (2), when $f(l).f(-l)$ is given, in which we find by expansion

$$\begin{aligned} A &= a^2+d^2+e^2, & D &= ef+d\,(a+b), \\ B &= b^2+f^2+d^2, & E &= fd+e\,(a+c), \\ C &= c^2+f^2+e^2, & F &= ed+f\,(b+c). \end{aligned}$$

With these values (2) becomes

$$l^6-Ll^4+Ml^2-N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where, if L, M, and N are essentially positive, then, according to Des Cartes' rule of signs, we must have an equation for l^2, and therefore for $f(l)$, whose roots cannot be of the form of $(l-p)^2=-y^2$, and therefore negative, but must be essentially *real.* The only question to be considered is, what is the essential sign of L, M, and N? In the expansion of (2), we shall find that the L of (3) is equal to

$$a^2+b^2+c^2+2f^2+2e^2+2d^2,$$

$$\begin{aligned} M = (ab-d^2)^2+(ac-e^2)^2+(bc-f^2)^2 \\ +2\,(af-ed)^2+2\,(be-f)^2+(cd-f)^2, \end{aligned}$$

and
$$N=\begin{vmatrix} a & d & e \\ d & b & f \\ e & f & c \end{vmatrix}$$

where L, M, N are, it is evident, essentially positive.

Ex. 3. It might be well to mention one peculiar case in the multiplication of determinants, as exhibiting or suggesting an easy treatment of a large number of theorems. It may be

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found in *Crelle*, Vols. 39 and 51; it is also given by Salmon and Brioschi.

Suppose $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

the equation to a conic, a, b the semi-axes.

If, now, we take any three points on the curve and form a triangle, its area could be expressed at once by the determinant given in Art. 8, in terms of the co-ordinates of its vertices; and similarly, in this case, the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \frac{y}{b} & \frac{y_1}{b} & \frac{y_2}{b} \\ \frac{x}{a} & \frac{x_1}{a} & \frac{x_2}{a} \end{vmatrix}$$

immediately suggests itself as expressing twice the area of the given triangle, $= \pm 2\,\frac{S}{ab}$, S being $=$ to the area of the triangle whose points are given (xy), (x_1y_1), (x_2y_2). If now we square this determinant, or multiply it by

$$\begin{vmatrix} \frac{x}{a} & \frac{y}{b} & -1 \\ \frac{x_1}{a} & \frac{y_1}{b} & -1 \\ \frac{x_2}{a} & \frac{y_2}{b} & -1 \end{vmatrix} = \mp 2\,\frac{S}{2ab},$$

we shall obtain a symmetrical determinant, as

$$\begin{vmatrix} a_1 & h & g \\ h & b_1 & f \\ g & f & c_1 \end{vmatrix} = -\frac{4S^2}{a^2b^2},$$

where $$S=\tfrac{1}{2}ab\,(-a_1b_1c_1+a_1f^2+b_1g^2+c_1h^2-2fgh)^{\frac{1}{2}};$$

and since, if the points are on the curve, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,\quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0,\ \text{ and }\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}-1=0,$$

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$$a_1 = b_1 = c_1 = 0, \text{ and likewise } S = \tfrac{1}{2}ab\,(-2fgh)^{\frac{1}{2}},$$

which is the value of the determinant

$$\begin{vmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{vmatrix}$$

where $h = \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1,\ g = \frac{xx_2}{a^2} + \frac{yy_2}{b^2} - 1, f = \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1,$

which can be reduced as follows:

Let c, d, e represent the sides of the triangle, and C, D, E the parallel semi-diameters respectively.

Then, from the nature of the ellipse, we have

$$\frac{c^2}{C^2} = \frac{(x_1-x_2)^2}{a^2} + \frac{(y_1-y_2)^2}{b^2}, \qquad \left\{ \begin{aligned} &\text{but } \frac{(x_1-x_2)^2}{a^2} + \frac{(y_1-y_2)^2}{b^2} \\ &\qquad = 2\left(1 - \frac{x_1x_2}{a_2} - \frac{y_1y_2}{b^2}\right), \end{aligned} \right.$$

$$\frac{d^2}{D^2} = \frac{(x_2-x)^2}{a^2} + \frac{(y_2-y)^2}{b^2},$$

$$\frac{e^2}{E^2} = \frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2},$$

and corresponding values for $\frac{d^2}{D^2}$ and $\frac{e^2}{E^2}$, which differ from the other values of $h, g,$ and f by only the factor 2 and the negative sign.

Therefore, by substituting, we have

$$\frac{4S^2}{a^2b^2} = -\begin{vmatrix} 0 & \frac{c^2}{2C^2} & \frac{d^2}{2D^2} \\ \frac{c^2}{2C^2} & 0 & \frac{e^2}{2E^2} \\ \frac{d^2}{2D^2} & \frac{e^2}{2E^2} & 0 \end{vmatrix} = \frac{c^2d^2e^2}{4C^2D^2E^2},$$

therefore $$S = \tfrac{1}{4}ab\,\frac{cde}{CDE}.$$

30. It must be borne in mind that the examples here given are simply for illustration, and to satisfy the reader that the

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principles employed are capable of wide application in all Co-ordinate Geometry.

Two theorems will now be added, which the reader will be able to prove in a manner more or less general.

1. *The square of a determinant of an even order can be expressed by a skew symmetric of an even order.*

2. *While a symmetric skew of an even order does not vanish, its inverse is a symmetric skew determinant.**

31. Let us now consider briefly *determinants arising from the roots of equations.*

It is well known that, by Sturm's theorem, we find the *number and places* of real roots—that the imaginary roots enter by pairs, and are equal in number to the variations of signs of the leading powers of x in all the functions.

Let $$\Delta = \pm \begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ c_1 & c_2 & \ldots\ldots & c_n \\ c_1^2 & c_2^2 & \ldots\ldots & c_n^2 \\ \ldots & & \ldots\ldots & \ldots \\ c_1^{n-1} & c_2^{n-1} & \ldots\ldots & c_n^{n-1} \end{vmatrix}$$

be the determinant formed from the roots of the equation

$$x^n + C_{n-1}x^{n-1} + C_{n-2}x^{n-2} + \ldots\ldots C_0 = 0.$$

Substituting c for x, we write

$$\begin{aligned} c_1^n + C_{n-1}c_1^{n-1} + C_{n-2}c_1^{n-2} \ldots C_0 &= 0, \\ c_2^n + C_{n-1}c_2^{n-1} + \&c. \qquad &= 0, \\ \&c. \qquad \&c. & \\ c_n^n + C_{n-1}c_n^{n-1} + C_{n-2}c_n^{n-2} + \ldots &= 0. \end{aligned}$$

* These theorems have, in fact, already been exhibited, but their applications to linear equations *generally* will be seen in *Crelle*, Vols. 51 and 52, and, for earlier investigations of the theory of Substitutions, see *Euler*, Vols. 15 and 20 of *Novi Commentarii Acad. Petrop.* Compare also the formulæ given by Rodigues in *Liouville*, tom. 5, with those of Euler here cited in *N. C. A. P.* under *De motu corporum rigidorum.*

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Let these equations be multiplied by any indeterminants, as $\kappa_1, \kappa_2 \ldots \kappa_n$, and assume

$$\kappa_1 c_1^s + \kappa_2 c_2^s + \kappa_3 c_3^s + \&c. = v \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

also

$$\kappa_1 + \kappa_2 + \ldots\ldots \kappa_n = 0,$$
$$\kappa_1 c_1 + \kappa_2 c_2 + \ldots\ldots \kappa_n c_n = 0,$$
$$\kappa_1 c_1^{n-1} + \kappa_2 c_2^{n-2} + \ldots\ldots \kappa_n c_n^{n-1} = 0;$$

whence, by a short algebraic process, we shall find

$$C_s = -\frac{1}{v}(\kappa_1 c_1^n + \kappa_2 c_2^n \ldots\ldots \kappa_n c_n^n) \ldots\ldots\ldots\ldots\ldots (2).$$

By differentiating the given determinant and employing the value of v, we have, from the determinant,

$$\frac{d\Delta}{dc_1^s} c_1^s + \frac{d\Delta}{dc_2^s} c_2^s + \ldots\ldots \frac{d\Delta}{dc_n^s} c_n^s = \Delta,$$

and, from (1),

$$\kappa_1 = \frac{d\Delta}{dc_1^s} \cdot \frac{v}{\Delta}, \quad \kappa_2 = \frac{d\Delta}{dc_2^s} \cdot \frac{v}{\Delta}, \quad \ldots\ldots \quad \kappa_n = \frac{d\Delta}{dc_n^s} \cdot \frac{v}{\Delta},$$

which values, substituted in (2), give

$$C_s = -\frac{1}{\Delta}\left(c_1^n \frac{d\Delta}{dc_1^s} + c_2^n \frac{d\Delta}{dc_2^s} \ldots + c_n^n \frac{d\Delta}{dc_n^s}\right),$$

which evidently represents the sums of the combinations of the roots taken $n-s$ and $n-s$.

Let us now seek for Δ in terms of the involved roots by their differences.

Let $$s = n-1,$$

and $$\phi(x) = (x-c_1)(x-c_2) \ldots (x-c_n);$$

and since κ_1, κ_2, &c. are any values

$$\kappa_1 = \frac{v}{\phi_1(c_1)}, \quad \kappa_2 = \frac{v}{\phi_1(c_2)}, \quad \ldots\ldots \quad \kappa_n = \frac{v}{\phi_1(c_n)},$$

therefore $$\frac{1}{\phi_1(c_1)} = \frac{1}{\Delta} \cdot \frac{d\Delta}{dc_1^{n-1}}, \quad \ldots\ldots \quad \frac{1}{\phi_1(c_n)} = \frac{1}{\Delta} \cdot \frac{d\Delta}{dc_n^{n-1}};$$

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but $$\frac{d\Delta}{dc_1^{n-1}} = \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ c_2 & c_3 & \cdots\cdots & c_n \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ c_2^{n-2} & c_3^{n-2} & \cdots\cdots & c_n^{n-2} \end{vmatrix} = \Delta_1.$$

Let $\phi'(x)$ designate

$$(x-c_2)(x-c_3)\ldots\ldots(x-c_n),$$

therefore $$\frac{1}{\phi_1'(c_2)} = \frac{1}{\Delta_1}\cdot\frac{d\Delta_1}{dc_2^{n-2}}.$$

Similarly, if we put

$$\frac{d\Delta_1}{dc_2^{n-2}} = \Delta_2, \quad \frac{d\Delta_2}{dc_3^{n-3}} = \Delta_3, \quad \ldots\ldots \quad \frac{d\Delta_{n-2}}{dc_{n-1}} = \Delta_{n-1} = 1,$$

and $\phi_2(x) = (x-c_3)\ldots(x-c_n)$, $\phi_3(x) = (x-c_4)\ldots(x-c_n)$,

$$\phi_{n-2}(x) = (x-c_{n-1})(x-c_n);$$

there will result

$$\frac{1}{\phi_2'(c_3)} = \frac{1}{\Delta_2}\cdot\frac{d\Delta_2}{dc_3^{n-3}}, \quad \ldots\ldots \quad \frac{1}{\phi_{n-2}'(c_{n-1})} = \frac{1}{\Delta_{n-1}};$$

whence, by multiplication, member by member,

$$\Delta = \phi'(c_1)\,\phi_1'(c_2)\,\phi_2'(c_3)\ldots\phi_{n-2}'(c_{n-1})$$
$$= (c_1-c_2)(c_1-c_3)\ldots(c_1-c_n)\ldots(c_2-c_n)\ldots(c_{n-1}-c_n) \ldots\ldots (3),$$

which is the product of the differences of n roots expressed as a determinant.

All this is easily generalized as follows:—

If, in $$\Delta = \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ c_1 & c_2 & \cdots\cdots & c_n \\ c_1^2 & c_2^2 & \cdots\cdots & c_n^2 \\ \cdots & \cdots & \cdots & \cdots \\ c_1^{n-1} & c_2^{n-1} & \cdots\cdots & c_n^{n-1} \end{vmatrix} \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

we consider that this determinant would vanish if $c_1 = c_2$, and that therefore $c_1 - c_2$ must be a factor, and what is true of these

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two roots is true of all the others considered two and two; hence we are enabled to write (3) at once.

Or we might prove true generally the method which is here exhibited as a special case,

$$\begin{vmatrix} 1 & 1 & 1 \\ c_1 & c_2 & c_3 \\ c_1^2 & c_2^2 & c_3^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ c_1-c_3 & c_2-c_3 & c_3 \\ c_1^2-c_3^2 & c_2^2-c_3^2 & c_3^2 \end{vmatrix}$$

$$= (c_1-c_3)(c_2-c_3)\begin{vmatrix} 1 & 1 \\ c_1+c_3 & c_2+c_3 \end{vmatrix} = (c_1-c_3)(c_2-c_3)(c_2-c_1).$$

Ex.: Prove that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ c_1 & c_2 & c_3 & c_4 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 \\ \cdots & \cdots & \cdots & \cdots \\ c_1^4 & c_2^4 & c_3^4 & c_4^4 \end{vmatrix} = (c_1-c_4)(c_2-c_4)(c_3-c_4)(c_2-c_3)(c_1-c_2)(c_1+c_2+c_3+c_4).$$

32. If now we proceed to form the square of (4) of the last Art., we may write the result

$$\begin{vmatrix} S_0 & S_1 & \cdots\cdots & S_{n-2} \\ S_1 & S_2 & \cdots\cdots & S_n \\ \cdots & \cdots & \cdots & \cdots \\ S_{n-1} & S_n & \cdots\cdots & S_{n-2} \end{vmatrix}$$

where S_0, S_2, ... S_n, &c. express the sum of the first, second, and nth powers of c_1, c_2, &c.

Thus, for example,

$$\begin{vmatrix} 1 & 1 \\ c_1 & c_2 \end{vmatrix}^2 = \begin{vmatrix} S_0 & S_1 \\ S_1 & S_2 \end{vmatrix} = (c_1-c_2)^2.$$

In the same manner we shall find

$$\begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_2 & S_3 & S_4 \end{vmatrix} = \Sigma(c_1-c_2)^2(c_2-c_3)^2(c_3-c_1)^2, \quad \&c. \qquad \&c.$$

where $\Sigma =$ the sum of the products.

These determinants are of great practical value in the theory of equations, inasmuch as, with their aid, as with the functions of Sturm, we determine the number of variations of signs, and,

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as stated at the beginning of the preceding Art., this determines the number of pairs of imaginary roots.

But if these determinants are all positive, there will be no variations, and consequently all the roots of the equation will be real.

To those acquainted with the general theory of equations these hints will be sufficient to show the bearing of determinants upon this subject; the real object in this and the preceding Art. being to prepare the way for the solution of linear differential equations by the use of the determinant notation.

33. *When $n-1$ particular integrals are given, to find the n^{th}.* *

Let us take the general linear differential equation, coefficients being constant

$$\frac{d^n y}{dx^n} + A\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots R\frac{dy}{dx} + Ty = 0\ldots\ldots\ldots\ldots(1).$$

If we separate the signs of operation from those of quantity, the part involving only signs of operation and constants may be considered as an operation performed on y, as

$$f\left(\frac{d}{dx}\right)y = 0.$$

From which we get y at once explicitly, if we are able to perform the inverse operation

$$\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}.$$

This we cannot easily do in its general form, but we can conceive the operation $f\left(\frac{d}{dx}\right)$ to be made up of certain binomial operations, and then perform the inverse operation for each of these. We will, however, in this case proceed in a different manner.

Let us first assume the n particular integrals, that is, values

* See Malmsten, in *Crelle*, vol. 39.

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that will satisfy (1), as $y_1, y_2 \ldots y_n$; coefficients now being variable.

Proceeding as in Art 31, and placing

$$\begin{aligned}
\kappa_1 y_1 + \kappa_2 y_2 \ldots\ldots \kappa_n y_n &= 0 \\
\kappa_1 y_1^1 + \kappa_2 y_2^1 \ldots\ldots \kappa_n y_n^1 &= 0 \\
\kappa_1 y_1^r + \kappa_2 y_2^r \ldots\ldots \kappa_n y_n^r &= v \\
\ldots \quad \ldots \quad \ldots \quad & \ldots \\
\kappa_1 y_1^{n-1} + \kappa_2 y_2^{n-1} \ldots\ldots \kappa_n y_n^{n-1} &= 0
\end{aligned}$$

we obtain

$$A_r = -\frac{1}{v}(\kappa_1 y_1^n + \kappa_2 y_2^n + \ldots \kappa_n y_n^n)^* \ldots\ldots\ldots\ldots (2).$$

Solving, as before, for the values of the indeterminates $\kappa_1, \kappa_2 \ldots \kappa_n$, and substituting in (2), we find, since

$$\Delta = \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_n \\ y_1^1 & y_2^1 & \ldots\ldots & y_n^1 \\ \ldots & & \ldots & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots\ldots & y_n^{n-1} \end{vmatrix}$$

that $$A_r = -\frac{1}{\Delta}\left(y_1^n \frac{d\Delta}{dy_1^r} + y_2^n \frac{d\Delta}{dy_2^r} + \ldots\ldots y_n^n \frac{d\Delta}{dy_n^r}\right).$$

In differentiating this determinant, we get Δ', or

$$\Delta' = y_1^n \frac{d\Delta}{dy_1^{n-1}} + y_2^n \frac{d\Delta}{dy_2^{n-1}} \ldots\ldots y_n^n \frac{d\Delta}{dy_n^{n-1}};$$

therefore $$A_r = -\frac{\Delta'}{\Delta} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Resuming now equation (1):

Let us suppose the $n-1$ particular integrals $y_1, y_2, \ldots y_{n-1}$ are known.†

Let
$$\left.\begin{aligned}
y &= y_1 \kappa_1 + y_2 \kappa_2 + \ldots\ldots y_{n-1} \kappa_{n-1} \\
0 &= y_1 \kappa_1' + y_2 \kappa_2' + \ldots\ldots y_{n-1} \kappa_{n-1}' \\
0 &= y_1^1 \kappa_1' + y_2^1 \kappa_2' + \ldots\ldots y_{n-1}^1 \kappa_{n-1}' \\
\ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
0 &= y_1^{n-3} \kappa_1' + y_2^{n-3} \kappa_2' + \ldots\ldots y_{n-1}^{n-3} \kappa_{n-1}'
\end{aligned}\right\} \ldots\ldots\ldots (5).$$

* $r, n, n-1$ do not, of course, indicate powers.

† *Crelle*, vol. 39, p. 94.

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Solving, we find

$$\kappa_1' : \kappa_2' : \ldots\ldots \kappa_{n-1}' :: \pm \begin{vmatrix} y_2 & y_3 & \ldots\ldots & y_{n-1} \\ y_2^1 & y_3^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_2^{n-3} & y_3^{n-3} & \ldots\ldots & y_{n-1}^{n-3} \end{vmatrix}$$

$$: \pm \begin{vmatrix} y_3 & y_4 & \ldots\ldots & y_1 \\ y_3^1 & y_4^1 & \ldots\ldots & y_1^1 \\ \ldots & \ldots & & \ldots \\ y_3^{n-3} & y_4^{n-3} & \ldots\ldots & y_1^{n-3} \end{vmatrix} : \ldots : \pm \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-2} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-2}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-3} & y_2^{n-3} & \ldots\ldots & y_{n-2}^{n-3} \end{vmatrix}$$

$$:: \Delta_1 : \Delta_2 : \ldots\ldots \Delta_{n-1}.$$

If now we differentiate successively equation (5), remembering the assumed relations between the $n-1$ functions, we shall have

$$\frac{dy}{dx} = y_1'\kappa_1 + y_2'\kappa_2 + \ldots\ldots y_{n-1}'\kappa_{n-1},$$

$$\frac{d^2y}{dx^2} = y_1''\kappa_1 + y_2''\kappa_2 + \ldots\ldots y_{n-1}''\kappa_{n-1},$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\frac{d^{n-1}y}{dx^{n-1}} = y_1^{n-1}\kappa_1 + y_2^{n-1}\kappa_2 + \ldots\ldots y_{n-1}^{n-1}\kappa_{n-1}$$
$$+ y_1^{n-2}\kappa_1' + y_2^{n-2}\kappa_2' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}',$$

$$\frac{d^ny}{dx^n} = y_1^n\kappa_1 + y_2^n\kappa_2 + \ldots\ldots y_{n-1}^n\kappa_{n-1}$$
$$+ 2\left(y_1^{n-1}\kappa_1' + y_2^{n-1}\kappa_2' + \ldots\ldots y_{n-1}^{n-1}\kappa_{n-1}'\right)$$
$$+ y_1^{n-2}\kappa_1'' + y_2^{n-2}\kappa_2'' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}'';$$

which, substituted in (1), give

$$y_1^{n-2}\kappa_1'' + y_2^{n-2}\kappa_2'' + \ldots\ldots y_{n-1}^{n-2}\kappa_{n-1}'' + \left[A\left(y_1^{n-2}\right) + 2y_1^{n-1}\right]\kappa_1'$$
$$+ \left[A\left(y_2^{n-2}\right) + 2y_2^{n-1}\right]\kappa_2' + \ldots\ldots \left[A\left(y_{n-1}^{n-2}\right) + 2y_{n-1}^{n-1}\right]\kappa_{n-1}' = 0.$$

Let $\qquad \kappa_1' = U\Delta_1, \quad \kappa_2' = U\Delta_2 \;\ldots\ldots\; \kappa_n' = U\Delta_n;$

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whence

$$\kappa_1'' = U'\Delta_1 + U\Delta_1', \quad \kappa_2'' = U'\Delta_2 + U\Delta_2' \ \ldots\ldots\ \kappa_n'' = U'\Delta_n + U\Delta_n'.$$

But $$y_1^{n-2}\Delta_1 + y_2^{n-2}\Delta_2 + \ldots\ldots y_{n-1}^{n-2}\Delta_{n-1}$$

furnishes the determinant

$$\begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-1} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-2} & y_2^{n-2} & \ldots\ldots & y_{n-1}^{n-2} \end{vmatrix} = \Delta,$$

and $$\frac{d\Delta}{dx} = \begin{vmatrix} y_1 & y_2 & \ldots\ldots & y_{n-1} \\ y_1^1 & y_2^1 & \ldots\ldots & y_{n-1}^1 \\ \ldots & \ldots & & \ldots \\ y_1^{n-1} & y_2^{n-1} & \ldots\ldots & y_{n-1}^{n-1} \end{vmatrix}$$

therefore $$y_1^{n-2}\Delta_1' + y_2^{n-2}\Delta_2' + \ldots\ldots y_{n-1}^{n-2}\Delta_{n-1}' = 0,$$

and $$y_1^{n-1}\Delta_1 - y_2^{n-1}\Delta_2 + \ldots\ldots y_{n-1}^{n-1}\Delta_{n-1} = \Delta';$$

therefore $$U'\Delta + UA\Delta + 2U\Delta' = 0,$$

$$\frac{U'}{U} + \frac{2\Delta'}{\Delta} + A = 0.$$

By integrating this equation, we have

$$U = \frac{e^{-\int A\,dx}}{\Delta^2};$$

or, substituting for U the values of κ_1 and Δ_1, and integrating again,

$$\kappa_s = \int \frac{\Delta_1}{\Delta^s} e^{-\int A\,dx}\,dx.$$

If we write $$\Delta = \frac{1}{\Delta} = \Sigma \pm y_1 \cdot y_2^1 \ldots\ldots y_{n-1}^{n-2},$$

we have $$\kappa_s = (-1)^{n-1}\int \frac{d\Delta}{dy_s^{n-2}} e^{-\int A\,dx}\,dx.$$

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Returning now to equation (4), we see that it can be written

$$A_r = \frac{\Delta'}{Ce^{-\int A\,dx}}, \text{ where } \Delta = Ce^{-\int A\,dx}.$$

A single instance is thus given in full, that the reader may judge for himself of the practical benefits of the determinant notation in conducting intricate analytical operations.

34. In the solution of simultaneous differential equations—that is, a system of equations with but one dependent variable, in which some form of this variable, as a function of the independent variables, must be found to satisfy all the equations—there is no reason why determinants may not be employed to effect the *elimination* (if this method be preferred to those of D'Alembert or Lagrange) as in the case of ordinary linear equations.

If, for example, we have three simultaneous differential equations of the form,

Ex. 1:
$$\frac{d}{dt}x + by + cz = 0,$$
$$ax + \frac{d}{dt}y + c'z = 0,$$
$$a'x + b'y + \frac{d}{dt}z = 0.$$

The condition of co-existence is the determinant

$$\begin{vmatrix} \frac{d}{dt} & b & c \\ a & \frac{d}{dt} & c' \\ a' & b' & \frac{d}{dt} \end{vmatrix} = \left[\frac{d}{dt}\left(\frac{d^2}{dt^2} - b'c'\right) + a\left(b'c - b\frac{d}{dt}\right) + a'\left(bc' - c\frac{d}{dt}\right)\right]x = 0;$$

i. e., $$\left[\frac{d^3}{dt^3} - (ab + a'c + b'c')\frac{d}{dt} + ab'c + a'bc'\right]x = 0,$$

and we can proceed to integrate at once this equation, giving rise to only three arbitrary constants.

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Ex. 2.—Let us take four simultaneous equations.

The equations of Airy, for determining the secular variations of the eccentricity and longitude of the perihelion, will serve as an illustration:

$$\frac{d}{dt}u + a_1 v - a_2 v' = 0,$$

$$a_1 u - \frac{d}{dt}v - a_2 u' = 0,$$

$$\frac{d}{dt}u' + b_1 v' - b_2 v = 0,$$

$$b_1 u' - \frac{d}{dt}v' - b_2 u = 0.$$

The determinant for eliminating the variables u, v, u', v', is therefore

$$\begin{vmatrix} \frac{d}{dt} & a_1 & -a_2 & 0 \\ a_1 & -\frac{d}{dt} & 0 & -a_2 \\ 0 & -b_2 & b_1 & \frac{d}{dt} \\ -b_2 & 0 & -\frac{d}{dt} & b_1 \end{vmatrix}$$

$$= \frac{d^4}{dt^4} + (a_1^2 + b_1^2 + 2a_2b_2)\frac{d^2}{dt^2} + (a_1b_1 - a_2b_2)^2 = 0;$$

which can readily be integrated, and is, of course, symmetrical for either of the variables. This may be regarded as the equation in u.

35. One other example upon this point, and then we shall proceed to another subject.

Suppose we have a pair of linear partial differential equations, as

$$\frac{dV}{dx}dx + \frac{dV}{dy}dy = 0,$$

$$\frac{dU}{dx}dx + \frac{dU}{dy}dy = 0,$$

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where U and V are functions of x and y; then the condition of the dependence of these functions is expressed by the determinant

$$\begin{vmatrix} \frac{dV}{dx} & \frac{dV}{dy} \\ \frac{dU}{dx} & \frac{dU}{dy} \end{vmatrix} = 0.$$

This leads us to the consideration of what are called *functional determinants;* and the general proposition is that, *when a functional determinant of a system of functions vanishes, it expresses the condition of dependence of the functions;* that is, we may test the dependence of functions in a manner analogous to that which we have employed to determine the co-existence of linear equations.

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CHAPTER III.

FUNCTIONAL DETERMINANTS.

36. As this subject is supposed to present some difficulties, and is of the highest interest in connection with geometrical researches, we shall seek in the first place to exhibit some of its principles in a very elementary form, and then proceed to show the field of application.

Suppose we have a series of functions $v_1, v_2 \ldots v_n$ of as many variables $x_1, x_2 \ldots x_n$, and by virtue of the relationship of these functions we are enabled to find

$$f(v_1 v_2 \ldots v_n) = 0,$$

in other words, that they are connected by an equation which vanishes identically. This relationship is expressible as a determinant* (Art. 35)

$$\begin{vmatrix} \frac{dv_1}{dx_1} & \frac{dv_1}{dx_2} & \cdots\cdots & \frac{dv_1}{dx_n} \\ \frac{dv_2}{dx_1} & \frac{dv_2}{dx_2} & \cdots\cdots & \frac{dv_2}{dx_n} \\ \cdots & \cdots & & \cdots \\ \frac{dv_n}{dx_1} & \frac{dv_n}{dx_2} & \cdots\cdots & \frac{dv_n}{dx_n} \end{vmatrix} = 0.$$

Then we say $v_1, v_2 \ldots v_n$ are *dependent*.

To fix our thoughts by an illustration, suppose

$$\begin{aligned} v_1 &= x + 2y + z, \\ v_2 &= x - 2y + 3z, \\ v_3 &= 2xy - xz + 4yz - 2z^2, \end{aligned}$$

* On this subject, see Jacobi, *Crelle*, Vol. 22. Spottiswoode, *Crelle*, Vol. 51. Pierce's *Analytical Mechanics*.

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then
$$\begin{vmatrix} \frac{dv_1}{dx} & \frac{dv_1}{dy} & \frac{dv_1}{dz} \\ \frac{dv_2}{dx} & \frac{dv_2}{dy} & \frac{dv_2}{dz} \\ \frac{dv_3}{dx} & \frac{dv_3}{dy} & \frac{dv_3}{dz} \end{vmatrix} = 0$$

becomes
$$xy \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix} = 0;$$

reducing, we find

$$-4(-x+4y-4z)+8(2y-z)-2(2x+4z)=0,$$

which, vanishing identically, shows the functions v_1, v_2, v_3 to be dependent.

37. Let us suppose the connecting equation to be

$$F(v_1 v_2 \ldots v_n) = 0.$$

If now we differentiate this equation in respect to any one of the functions $v_1, v_2 \ldots v_n$ under consideration, regarded as functions of the variables $x_1, x_2 \ldots x_n$, we must have, in general,*

$$\frac{dF}{dv_r} = \frac{dF}{dx_1}\cdot\frac{dx_1}{dv_r} + \frac{dF}{dx_2}\cdot\frac{dx_2}{dv_r} \;\ldots\ldots\; \frac{dF}{dx_n}\cdot\frac{dx_n}{dv_r};$$

when $F = v_r$ the left member of this equation $= 1$. And, in general, if we replace F by v_s, we shall have, when $s = r$,

$$1 = \frac{dv_s}{dx_1}\cdot\frac{dx_1}{dv_s} + \frac{dv_s}{dx_2}\cdot\frac{dx_2}{dv_s} \;\ldots\ldots\; \frac{dv_s}{dx_n}\cdot\frac{dx_n}{dv_s} \;\ldots\ldots\ldots\; (1).$$

If, however, s is not equal to r, we must have

$$0 = \frac{dv_s}{dx_1}\cdot\frac{dx_1}{dv_r} + \frac{dv_s}{dx_2}\cdot\frac{dx_2}{dv_r} \;\ldots\ldots\; \frac{dv_s}{dx_n}\cdot\frac{dx_n}{dv_r} \;\ldots\ldots\ldots\; (2).$$

* Jacobi in *Crelle*, Vol. 22.

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In the same manner,

$$\frac{dx_r}{dv_1}\cdot\frac{dv_1}{dx_r}+\ldots\ldots\frac{dx_r}{dv_n}\cdot\frac{dv_n}{dx_r}=1 \;\ldots\ldots\ldots\ldots\; (3),$$

$$\frac{dx_r}{dv_1}\cdot\frac{dv_1}{dx_s}+\ldots\ldots\frac{dx_r}{dv_n}\cdot\frac{dv_n}{dx_s}=0 \;\ldots\ldots\ldots\ldots\; (4).$$

By means of (1), (2), (3), (4), we are enabled to solve a system of equations analogous to the following:—

$$\left.\begin{array}{l} y\frac{dv}{dx}+y_1\frac{dv}{dx_1}\;\ldots\ldots\; y_n\frac{dv}{dx_n}=u \\ y\frac{dv_1}{dx}+y_1\frac{dv_1}{dx_1}\;\ldots\ldots\; y_n\frac{dv_1}{dx_n}=u_1 \\ \ldots\quad\ldots\quad\ldots\quad\ldots \\ y\frac{dv_n}{dx}+y_1\frac{dv_n}{dx_1}\;\ldots\ldots\; y_n\frac{dv_n}{dx_n}=u_n \end{array}\right\}\;\ldots\ldots\ldots\ldots\; (5).$$

If we multiply these equations by

$$\frac{dx}{dv},\ \frac{dx}{dv_1}\;\ldots\ldots\;\frac{dx}{dv_n},\ \text{\&c. \&c.},$$

and add, we shall have, by virtue of (3) and (4),

$$\left.\begin{array}{l} y\;=u\frac{dx}{dv}+u_1\frac{dx}{dv_1}\;\ldots\ldots\; u_n\frac{dx}{dv_n} \\ y_1=u\frac{dx_1}{dv}+u_1\frac{dx_1}{dv_1}\;\ldots\ldots\; u_n\frac{dx_1}{dv_n} \\ \ldots\quad\ldots\quad\ldots\quad\ldots \\ y_n=u\frac{dx_n}{dv}+u_1\frac{dx_n}{dv_1}\;\ldots\ldots\; u_n\frac{dx_n}{dv_n} \end{array}\right\}\;\ldots\ldots\ldots\ldots\; (6).$$

It is evident that, if the given functions $v, \ldots v_n$ are independent, and $y=y_1=y_n=0$, then $u=u_1=u_n=0$; or, in other words, if the given functions are independent, then (5) and (6) reduce in turn to 0.

38. The question now arises, how shall we express the solution of such systems as the above in the ordinary language

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of determinants? If we examine the solution of system (5) of the preceding Art., we shall see that what might be denominated the modulus of transformation is the determinant

$$\Delta = \begin{vmatrix} \frac{dv}{dx} \cdot \frac{dv}{dx_1} \ldots\ldots \frac{dv}{dx_n} \\ \frac{dv_1}{dx} \cdot \frac{dv_1}{dx_1} \ldots\ldots \frac{dv_1}{dx_n} \\ \ldots \quad \ldots \quad \ldots \\ \frac{dv_n}{dx} \cdot \frac{dv_n}{dx_1} \ldots\ldots \frac{dv_n}{dx_n} \end{vmatrix}$$

or

$$\begin{vmatrix} \Delta\frac{dx}{dv} & \Delta\frac{dx_1}{dv} & \ldots\ldots & \Delta\frac{dx_n}{dv} \\ \Delta'\frac{dx}{dv_1} & \Delta\frac{dx_1}{dv_1} & \ldots\ldots & \Delta\frac{dx_n}{dv_1} \\ \ldots & \ldots & \ldots & \ldots \\ \Delta\frac{dx}{dv_n} & \Delta\frac{dx_1}{dv_n} & \ldots\ldots & \Delta\frac{dx_n}{dv_n} \end{vmatrix} = \Delta^n,$$

since, manifestly, writing

$$\begin{vmatrix} \frac{dx}{dv} \cdot \frac{dx}{dv_1} \ldots\ldots \frac{dx}{dv_n} \\ \ldots \quad \ldots \quad \ldots, \\ \ldots \quad \ldots \quad \ldots \\ \frac{dx_n}{dv} \cdot \frac{dx_n}{dv_1} \ldots\ldots \frac{dx_n}{dv_n} \end{vmatrix} = \Delta',$$

we must have $$\Delta \times \Delta' = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Hence the notation to be adopted, which is all that is required, is sufficiently evident. If now we differentiate Δ' in respect to any one of its constituents, as $\frac{dx_i}{dv_k}$, we shall have $\frac{d\Delta'}{d\frac{dx_i}{dv_k}} = A$;

but, in consequence of (1), we are enabled to write $A = \Delta'\frac{dv_k}{dx_i}$, where $A =$ the corresponding minor, and therefore

$$\frac{d\Delta'}{d\frac{dx_i}{dv_k}} = \Delta' \frac{dv_k}{dx_i}.$$

In the same manner, in general,

$$\frac{d\Delta}{d\frac{dv_i}{dx_k}} = \Delta\frac{dx_k}{dv_i}, \quad \text{therefore} \quad \frac{d\Delta'}{d\frac{dx_i}{dv_k}} \cdot \frac{d\Delta}{d\frac{dv_i}{dx_k}} = \frac{dv_k}{dx_i} \cdot \frac{dx_k}{dv_i}.$$

The same course of reasoning may be applied to the connecting equations. That is, if

$$f_1 = 0 \; \ldots\ldots \; f_n = 0$$

connect the variables $x_1, x_2 \ldots x_n$ with $v_1, v_2 \ldots v_n$; then, inversely, if we find from $f_1 = 0$, &c., the values of $v_1, v_2 \ldots v_n$, and substitute these in the same equations, these will vanish identically; or, since we may eliminate $n-1$ of the variables from these equations, each may be treated as the function of a single variable and the given functions; therefore

$$\frac{df_k}{dv_1} \cdot \frac{dv_1}{dx_k} + \frac{df_k}{dv_2} \cdot \frac{dv_2}{dx_k} \;\ldots\ldots\; \frac{df_k}{dv_n} \cdot \frac{dv_n}{dx_k} = \frac{df_k}{dx_k}.$$

If $k = 1, 2 \ldots n$, we shall obtain n equations, from which, eliminating the differentials, a linear partial differential equation will arise, which shall be satisfied by the primitive equation under consideration, as $f_k = 0$.

Proceeding in a manner similar to that for obtaining (1), we write, finally,

$$\Delta\Sigma\left(\pm\frac{df_1}{dv_1}\frac{df_2}{dv_2} \;\ldots\ldots\; \frac{df_n}{dv_n}\right) = (-1)^n\,\Sigma\left(\pm\frac{df_1}{dx_1}\frac{df_2}{dx_2} \;\ldots\ldots\; \frac{df_n}{dx_n}\right).$$

The general application of these principles to the trans-

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formation of multiple integrals, as

$$\int^n V\,dv_1\,dv_2 \,\ldots\ldots\, dv_n,$$

where the functions $v_1 \ldots v_n$ are connected with the same number of other variables $x_1 \ldots x_n$ by equations similar to those assumed above, will not be considered.

It may, however, be remarked that, in transforming from one set of variables to another, the formula of transformation

$$\int^n V\,dv_1 \ldots\ldots dv_n = \int^n V\frac{df_1}{dx_1} \,\ldots\ldots\, \frac{df_n}{dx_n}\,.\,dx_1 \ldots\ldots dx_n$$

reduces at once to

$$\int^n V\,dv_1 \ldots\ldots dv_n = \int^n V\,.\,\Delta\,dx_1 \ldots\ldots dx_n,$$

and $$\int^n V\,dx_1 \ldots\ldots dx_n = \int^n V\,.\,\Delta'\,dv_1 \ldots\ldots dv_n.$$

On this subject, see Baltzer, p. 64.

39. *The Jacobian.*—The determinant already considered,

$$\begin{vmatrix} \dfrac{dU_1}{dx_1} & \cdots\cdots & \dfrac{dU_1}{dx_n} \\ \cdots & \cdots & \cdots \\ \dfrac{dU_n}{dx_1} & \cdots\cdots & \dfrac{dU_n}{dx_n} \end{vmatrix}$$

which, after Jacobi, is called the Jacobian, and generally denoted by J, has received considerable attention in the theory of elimination. The principal proposition is that, if a system of homogeneous equations be satisfied by a set of values, these values will satisfy both the Jacobian and its differential in regard to all the variables.

Let us take a system of three equations

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0;$$

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J will then be written

$$\begin{vmatrix} \frac{du_1}{dx} & \frac{du_1}{dy} & \frac{du_1}{dz} \\ \cdots & \cdots & \cdots \\ \frac{du_3}{dx} & \frac{du_3}{dy} & \frac{du_3}{dz} \end{vmatrix} = \Delta = J;$$

and let us assume, what is not difficult to prove, that

$$x\frac{du_1}{dx} + y\frac{du_1}{dy} + z\frac{du_1}{dz} = au_1,$$

$$x\frac{du_2}{dx} + y\frac{du_2}{dy} + z\frac{du_2}{dz} = au_2,$$

$$x\frac{du_3}{dx} + y\frac{du_3}{dy} + z\frac{du_3}{dz} = au_3.$$

Solving for x, we have, Art. 12,

$$(1) \ldots\ldots \Delta x = U_1 au_1 + U_2 au_2 + U_3 au_3,$$

where U_1, U_2, &c. = the minors. We see here that, if u_1, u_2, u_3 vanish, the determinant vanishes.

Differentiating (1) in respect to x and y,

$$(2) \ldots\ldots \Delta + x\frac{d\Delta}{dx} = au_1\frac{dU_1}{dx} + au_2\frac{dU_2}{dx} + au_3\frac{dU_3}{dx} + a\left(\frac{du_1}{dx}U_1 + \frac{du_2}{dx}U_2 + \frac{du_3}{dx}U_3\right),$$

$$(3) \ldots\ldots x\frac{d\Delta}{dy} = au_1\frac{dU_1}{dy} + au_2\frac{dU_2}{dy} + au_3\frac{dU_3}{dy} + a\left(\frac{du_1}{dy}U_1 + \frac{du_2}{dy}U_2 + \frac{du_3}{dy}U_3\right).$$

But the first parenthesis = Δ, Art. 12, and the second parenthesis = 0.

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Again, introducing the supposition $u_1 = u_2 = u_3 = 0$, we see that $\frac{d\Delta}{dy}$ and $\frac{d\Delta}{dx}$ must vanish, since (2) and (3) in this case, in consequence of (1), reduce to 0.

The application of this principle is obvious; for if we have three equations homogeneous in the second degree, their J will be of the third, and each of its differentials of the second, and these three new equations will be satisfied by the values common to the given equations. We have then

$$u_1 = 0, \quad \frac{d\Delta}{x} = 0,$$

$$u_2 = 0, \quad \frac{d\Delta}{y} = 0,$$

$$u_3 = 0, \quad \frac{d\Delta}{z} = 0,$$

to eliminate x^2, y^2, z^2, xy, zy, xz; and therefore the *eliminant*, that is, the eliminating determinant, can be formed.

When the given equations are of the third degree homogeneous, J is of the sixth, its differentials of the fifth; and by using Sylvester's dialytic process, we can eliminate the twenty-one quantities of an equation of the fifth degree.

40. *The Hessian.*—We will now show how to form this important determinant. Let V be any homogeneous function of n variables, analogous to $(a, b, c, d \between x, y)^3$, and, taking its second differential coefficients in respect to each of the variables, we write, for the special case,

$$H = \begin{vmatrix} ax+by & bx+cy \\ bx+cy & cx+dy \end{vmatrix}$$

This is called the Hessian, after the late Dr. Otto Hesse, of Munich.

The degree of the determinant will be $n(p-2)$, where $p =$ the degree of the function, and n the number of variables.

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If we connect the variables x and y with two others u and z by the equations

$$x = eu + fz,$$
$$y = e_1 u + f_1 z,$$

calling the transformed function V', and taking its second differentials, and indicating the Hessian thus formed by H' we may write

$$H' = H \times \Delta^2,$$

where

$$\Delta = \begin{vmatrix} e & f \\ e_1 & f_1 \end{vmatrix}$$

In orthogonal substitutions, $\Delta^2 = 1$, and $H = H'$.

Hesse has shown, in the use of this theorem, that if $V = 0$ be an equation to a plane curve of the nth order, the vanishing of the Hessian indicates the condition by which the curve reduces to a pencil of n right lines; and in like manner, if $V = 0$ be an equation to a surface, this surface reduces to a cone when H vanishes.*

* *Crelle*, vol. 42, p. 123.

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CHAPTER IV.

SOME APPLICATIONS.

41. In proceeding to the common applications of what has been explained, it will be necessary to introduce some of the terms of higher algebra; thus,

$$(a, b, c \between x, y)^2$$

is called a binary quadratic, which, written fully, is simply

$$ax^2 + 2bxy + cy^2,$$

and since it is a homogeneous function it is called also a *quantic*. If any quantic is to be considered apart from numerical coefficients, it is written

$$(a, b, c \between x, y)^n.$$

Let us now take the first expression, and linearly transform it, substituting
$$x = lx + my,$$
$$y = l'x + m'y,$$
and we shall have $$Ax^2 + 2Bxy + Cy^2$$
as the transformed function. If, now, we compare the Hessian of the given and the transformed expression, we shall find the relation given in the last Art. to be true, viz.,

$$H' = H \,.\, \Delta^2,$$

or, in full, $$AC - B^2 = (ac - b^2)\,(lm' - l'm)^2.$$

Now H and H' are called respectively the *discriminants* of the given and transformed quadratic.

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42. When a quantic has been transformed as above, any function of its coefficients is called an *invariant*. Hence $ac-b^2$ is also, by definition, an invariant; and, in general, a quantic of the quadratic class, irrespective of its variables, has no other invariant than its own discriminant, and in such cases the two terms indicate identical functions. Now, when we take the Hessian of any quantic, or what is sometimes called the second emanant, we obtain the *covariant* of the quantic, that is, a function of the coefficients involving the variables of the given quantic.

43. *Study first.*—Let us write the quadric surface

$$ax^2+by^2+cz^2+2exy+2fxz+2hyz+2gx+2iy+2kz+d=0;$$

the discriminant will then be

$$\begin{vmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & k \\ g & i & k & d \end{vmatrix} = 0 \dots\dots\dots\dots\dots\dots (1),$$

which may be formed in the manner already described, or we may transform to any parallel axes drawn through $x'y'z'$ by writing $x+x'$ for x, &c., and we shall find certain relations connecting the coefficients a, b, &c. with a', b', &c.; in other words, that there are functions of the given coefficients equal to the same functions of the transformed coefficients.

By taking the differentials in respect to each of the variables, we shall find the new coefficient of x to be

$$2\,(ax'+ey'+fz'+g),$$

and the condition that this general equation shall represent a cone will be the determinant of the following equations,

$$\left.\begin{aligned} ax'+ey'+fz'+g &= 0 \\ ex'+by'+hz'+i &= 0 \\ fx'+hy'+cz'+k &= 0 \\ gx'+iy'+kz'+d &= 0 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots (2).$$

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The determinant of which is the same as (1); the coordinates of the new vertex satisfying each of the above equations.

Forming now the first minors of this determinant, we have

$$a\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} - e\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} + f\begin{vmatrix} i & e & b \\ k & f & h \\ d & g & i \end{vmatrix} - g\begin{vmatrix} e & b & h \\ f & h & c \\ g & i & k \end{vmatrix}$$

and the second minors

$$ab\begin{vmatrix} c & k \\ k & d \end{vmatrix} + ah\begin{vmatrix} k & h \\ d & i \end{vmatrix} + ai\begin{vmatrix} h & c \\ i & k \end{vmatrix} \text{ \&c.}$$

Considering the first and second minors, we see that

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} \times b = \begin{vmatrix} b & i \\ i & d \end{vmatrix} \times \begin{vmatrix} b & h \\ h & c \end{vmatrix} - \begin{vmatrix} b & h \\ i & k \end{vmatrix}^2,$$

since

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} \times b + h^2 i^2 - h^2 i^2 = (bc - h^2)(bd - i^2) - (bk - hi)^2;$$

and we shall find, in general, that any first minor, multiplied by a constituent, is expressible in terms of the second minors, formed from this first minor.

Thus we shall find E; *i.e.*, second first minor, or

$$\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} \times c + fhk^2 - fhk^2$$
$$= \begin{vmatrix} e & f \\ h & c \end{vmatrix} \cdot \begin{vmatrix} c & k \\ k & d \end{vmatrix} - \begin{vmatrix} h & c \\ i & k \end{vmatrix} \cdot \begin{vmatrix} f & c \\ g & k \end{vmatrix}$$

Also A, or

$$\begin{vmatrix} b & h & i \\ h & c & k \\ i & k & d \end{vmatrix} . d = \begin{vmatrix} c & k \\ k & d \end{vmatrix} \cdot \begin{vmatrix} b & i \\ i & d \end{vmatrix} - \begin{vmatrix} h & k \\ i & d \end{vmatrix}^2 \quad \ldots\ldots (3),$$

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and

$$\begin{vmatrix} h & i & e \\ c & k & f \\ k & d & g \end{vmatrix} . d + * - * = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} c & k \\ k & d \end{vmatrix} - \begin{vmatrix} h & k \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} \quad \text{.........(4)}.$$

If, now, A and $E = 0$, (4) becomes

$$\begin{vmatrix} h & k \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} c & k \\ k & d \end{vmatrix}$$

and (3) reduces to $\begin{vmatrix} c & k \\ k & d \end{vmatrix} . \begin{vmatrix} b & i \\ i & d \end{vmatrix} = \begin{vmatrix} h & k \\ i & d \end{vmatrix}^2$

And these, multiplied together, give

$$\begin{vmatrix} b & i \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} = \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} h & k \\ i & d \end{vmatrix}$$

but minor F, *i. e.*,

$$\begin{vmatrix} e & b & i \\ f & h & k \\ g & i & d \end{vmatrix} . d + * - * = \begin{vmatrix} b & i \\ i & d \end{vmatrix} . \begin{vmatrix} f & k \\ g & d \end{vmatrix} - \begin{vmatrix} e & g \\ i & d \end{vmatrix} . \begin{vmatrix} h & k \\ i & d \end{vmatrix}$$

hence, on the supposition that $A = E = 0$, we have $F = 0$.

Continuing our analysis, we find that, when $A = 0 = E$, or $E = F = 0$, we have $\Delta = 0$.*

In general, we write the following, analogous to (2) and (3),

$$c\frac{d\Delta}{da} = \frac{d^2\Delta}{da\,db} \cdot \frac{d^2\Delta}{da\,dd} - \left(\frac{d^2\Delta}{da\,di}\right)^2,$$

$$c\frac{d\Delta}{de} = \frac{d^2\Delta}{dd\,de} \cdot \frac{d^2\Delta}{da\,db} - \frac{d^2\Delta}{da\,di} \cdot \frac{d^2\Delta}{db\,dg},$$

$$b\frac{d\Delta}{df} = \frac{d^2\Delta}{dd\,df} \cdot \frac{d^2\Delta}{da\,dc} - \frac{d^2\Delta}{da\,dk} \cdot \frac{d^2\Delta}{dc\,dg}.$$

* For an extension of the geometrical applications, herein considered, to tangential coordinates, and the determination of circular sections, see *Philosophical Mag.*, vol. xiv., 4th series, p. 393.

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Assume
$$\left.\begin{aligned} eE - fF + gG &= 0 \\ hE - cF + kG &= 0 \\ iE - kF + dG &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

which gives the determinant for the elimination of G and F,

$$\begin{vmatrix} e & -f & g \\ h & -c & k \\ i & -k & d \end{vmatrix} E = E \,.\, E = 0\,; \;\; i.e., \;\; E = 0\,;$$

therefore
$$F = 0 \;\text{ and }\; G = 0.$$

By equations similar to (5), as

$$aA + fF - gG = 0,$$
$$\&c. \qquad \&c.,$$

we may show that $AB = 0$, from which it follows that, if $B = 0$, $H = I = 0$; or, when $A = 0$, $F = G = 0$.

It can be shown that equations (5) are true when $\Delta = 0$ and $A = 0$ in all cases.

Let us now, in view of these suppositions and results, consider the nature of the surface given at the head of this Article.

Suppose $\dfrac{d^2\Delta}{da\,dd} = 0$, $\dfrac{d^2\Delta}{db\,dd} = 0$, and $-\dfrac{d^2\Delta}{dc\,dd} = 0$,

and consequently $\dfrac{d^2\Delta}{dd\,dh} = 0$; and suppose also a to be negative; then, multiplying the surface by a, subtract $(ey + fz + g)^2$, and finally let $ax' = ax + ey + fz + g$, and we shall have

$$a^2x'^2 + 2\,\frac{d^2\Delta}{dc\,di}\,y + 2\,\frac{d^2\Delta}{db\,dk}\,z + \frac{d^2\Delta}{db\,dc} = 0,$$

an equation to a *parabolic cylinder*.

The three latter suppositions applied to the surface reduce it to

$$a^2x'^2 + \frac{d^2\Delta}{dc\,dd}\,y^2 + 2\left(\frac{d^2\Delta}{dd\,dh}\,z + \frac{d^2\Delta}{dc\,di}\right)y + \frac{d^2\Delta}{db\,dd}\,z^2$$
$$+ 2\,\frac{d^2\Delta}{db\,dk}\,z + \frac{d^2\Delta}{db\,dc} = 0.$$

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This equation, multiplied by $\frac{d^2\Delta}{dc\,dd}$, adding and subtracting $\left(\frac{d^2\Delta}{dd\,dh}z + \frac{d^2\Delta}{dc\,di}\right)^2$, and finally making

$$\frac{d^2\Delta}{dc\,dd}y' = \frac{d^2\Delta}{dc\,dd}y + \frac{d^2\Delta}{dd\,dh}z + \frac{d^2\Delta}{dc\,di},$$

we obtain

$$a^2\frac{d^2\Delta}{dc\,dd}x'^2 + \left(\frac{d^2\Delta}{dc\,dd}\right)^2 y'^2 + a\left(\frac{d\Delta}{dd}z^2 + 2\frac{d\Delta}{dk}z + \frac{d\Delta}{dc}\right) = 0.$$

If, now, we multiply this by $\frac{d\Delta}{dd}$, add and subtract $a\left(\frac{d\Delta}{dk}\right)^2$, and put $\frac{d\Delta}{dd}z$ for $\frac{d\Delta}{dd}z + \frac{d\Delta}{dk}$, we have

$$a^2\frac{d^2\Delta}{dc\,dd}\,\frac{d\Delta}{dd}x'^2 + \left(\frac{d^2\Delta}{dc\,dd}\right)^2\frac{d\Delta}{dd}y'^2 + a\left(\frac{d\Delta}{dd}\right)^2 z'^2 + a\frac{d^2\Delta}{dc\,dd}\Delta = 0,$$

which is the equation to an *ellipsoid* when Δ is negative, and $\frac{d\Delta}{dd}$, $\frac{d^2\Delta}{dc\,dd}$ both positive.

When Δ is positive, and $\frac{d\Delta}{dd}$, $\frac{d^2\Delta}{dc\,dd}$ either one or both negative, this equation represents a *hyperboloid of one sheet.* If Δ be negative it represents a *hyperboloid of two sheets*, if it vanishes a *cone.* Also, if $\frac{d\Delta}{dd} = 0$, it is the equation to an *elliptic* or *hyperbolic paraboloid*, according as $\frac{d^2\Delta}{dc\,dd}$ is positive or negative. In the same manner it represents an *elliptic* or *hyperbolic cylinder* when $\frac{d\Delta}{dd} = \frac{d\Delta}{dk} = 0$, and $\frac{d^2\Delta}{dc\,dd}$ is positive or negative.

To find the plane perpendicular to the chord to which it is conjugate; *i.e.*, the diametral plane.

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Let l, m, n be the direction cosines of the chord, the plane in question will be, from equations (1),

$$l(ax+ey+fz+g)+m(ex+by+hz+i)+n(fx+hy+cz+k)=0,$$

provided l, m, n are proportional to the coefficients of the variables x, y, z; and we shall have, in that case,

$$\left.\begin{array}{l} la+me+nf=pl \\ le+mb+nh=pm \\ lc+mh+nc=pn \end{array}\right\} \dots\dots\dots\dots\dots\dots\dots (6);$$

therefore, to find p, we have the determinant

$$\begin{vmatrix} a-p & e & f \\ e & b-p & h \\ c & h & c-p \end{vmatrix}$$

The value of p being found and substituted in equations (6), we shall obtain the values of l, m, n, and thus be able to find *the three diametral planes of the surface.*

We conclude this study with the simple remark that we are not here concerned with teaching Modern Geometry, but with an exercise for the practice of Determinants, and to indicate their use in the investigation of *loci.*

44. The Jacobian, which has already been described, deserves, on account of its importance, a special consideration.

Study second.—(*a*) Let V and V_1 be two functions, homogeneous in the second degree, J is then

$$\begin{vmatrix} \dfrac{dV}{dx} & \dfrac{dV}{dy} \\ \dfrac{dV_1}{dx} & \dfrac{dV_1}{dy} \end{vmatrix} = 0,$$

which, under the conditions mentioned, determines the foci of involution of two pairs of points.

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(*b*) Let $S_1=0$, $S_2=0$, $S_3=0$ be three circles, and $u=0$ the equation to the circle orthotomic. The polar of any point on $u\ (xyz)$, with regard to each of the given circles, will pass through a single point. Let u_1, u_2, u_3, &c. represent the differentials $\frac{du}{dx}$, &c., then

$$lu_1+mu_2+nu_3 = 0,$$

$$lv_1+mv_2+nv_3 = 0,$$

$$lw_1+mw_2+nw_3 = 0.$$

The determinant of which is a Jacobian and $=u$, the equation of the circle orthotomic required.

(*c*) If we proceed to the conicoids, as V, V_1, V_2, the equations of the three polars will be

$$V'x+V''y+V'''z = 0,$$

$$V_1'x+V_1''y+V_1'''z = 0,$$

$$V_2'x+V_2''y+V_2'''z = 0;$$

therefore

$$\begin{vmatrix} \frac{dV'}{dx} & \frac{dV''}{dy} & \frac{dV'''}{dz} \\ \frac{dV_1'}{dx} & \frac{dV_1''}{dy} & \frac{dV_1'''}{dz} \\ \frac{dV_2'}{dx} & \frac{dV_2''}{dy} & \frac{dV_2'''}{dz} \end{vmatrix} = J,$$

and is a curve of the third order, in other words, the locus of a point whose polars, in regard to V, V_1, V_2, meet in a point.

(*d*) It is easily shown that two conics always intersect in four points.

Let V and V_1 intersect, and through these points draw V_2. Then the J of the three conics is the equation to the curve which cuts V_2 *in six points.*

(*e*) If we form the Hessian of $lV+mV_1+nV_2$; then, if we

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examine the coefficients of l, m, n, we shall find them invariants of V, V_1, V_2, one of which vanishes whenever $lV+mV_1+nV_2$ represents two planes; the other vanishes (as shown by Prof. Cayley) when any two of the eight points of intersection coincide, and their J is a curve of the sixth order, when V, V_1, V_2 represent quadrics, and this curve is the locus of a point whose polar planes meet in a line.

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