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ART. I.—*The Spirit of the Fathers of Western Presbyterianism.*

ON Tuesday, February 12th, of the present year, a centenary convention was held at Pittsburgh, Pennsylvania, composed of representatives of the twenty Presbyteries contained in the four Synods of Pittsburgh, Allegheny, Wheeling, and Ohio, which was designed to commemorate the visit to that region of the Rev. Charles Beatty and the Rev. George Duffield, by the appointment of the old "Synod of New York and Philadelphia." While the interest in the religious history of that region, so important in itself and in its influence upon the Presbyterian Church, is fresh, it is a favourable time to consider some points in the character and labours of its pioneer ministers.

It may be premised that this is a late hour to hold a "centenary" convention. The visit of Messrs. Beatty and Duffield was made in the summer of 1766; and the commemoration of that event is a year too late. But we cannot grant that to have been the kindling of the light of Presbyterianism in that territory. In the early part of the last century large numbers of the people from the North of Ireland were driven by the are therefore negatively Messianic. Ecclesiastes sets forth the unsatisfactory nature of all the splendour even of Solomon, when enjoyed without God. And the book of Lamentations at once completes the series and links this with the lessons of the succeeding period by bewailing its overthrow in consequence of its ungodliness, a result which it required centuries to develope.

To sum up the results at which we have arrived. The Psalms unfold the doctrine of the Messiah for the most part consciously and from the human side. They portray him as the man raised to sovereignty over the universe, as the righteous sufferer whose unparalleled sorrows result in the salvation of the world, as the triumphant monarch who subdues all opposition, rules peacefully over the whole world and to the end of time, is wedded to his people in holy love (an idea expanded likewise in the Song of Solomon) and who is a priest as well as a king. The other poetical books develope the doctrine of the Messiah for the most part unconsciously and from the Divine side. He is the Wisdom of God celebrated in Proverbs, the Redeemer in whom Job declared his confidence, the founder of an empire which has neither the unsatisfactory nature of worldly grandeur set forth in Ecclesiastes, nor its transitory character as shown in the Lamentations.

Art. V.—*The Philosophy of Mathematics.*

Whilst there are few who have not some knowledge of this science, fewer have ever asked themselves, What is Mathematics? and when the question is proposed a less number still are able to give a satisfactory answer. Unlike most other sciences, the name of this is not distinctive. Mathematics—τα μαθηματα—literally means, *things to be learned.* Accordingly, when the Greeks used the expression in a technical sense, they meant all the then known sciences. The subsequent use of the word in the restricted sense in which it is now always employed, is arbitrary, except so far as this usage may be justified by the fact that the particular science to which it is appropriated lies at the basis of all physical science.

Another reason doubtless why so few have a clear conception of mathematics as a well-defined science is, that the term is a plural. This would seem to imply that it denoted, not a single science, but a number of sciences, analogous yet distinct and independent—in some respects similar yet without any logical connection—rods of the same bundle rather than branches of one vine.

Moreover, the several branches of mathematics do actually differ greatly from each other—so much so that an individual may be thoroughly familiar with one branch and yet entirely ignorant of even the elements of others. A higher division of this science is not always a mere extension of a lower. The distinction between some at least is a difference not merely in degree, but in kind. It is not without meaning, therefore, that the more advanced branches are called, not merely higher, but transcendental. The same mathematical problem may be solved by entirely different methods, each involving ideas and processes peculiar to itself. In the study of different branches of mathematics entirely different faculties of the mind are exercised, so that not unfrequently the same individual may master with ease certain branches, and yet have no aptitude for others.

For the reasons mentioned, not only is the conception which most have of mathematics as a science vague and unsatisfactory, but the fact is, the true idea of the science as a systematic whole and the precise logical relation of the several parts, were not until within our own day, even by the mathematicians themselves, accurately determined.

Comte's great work, his "Cours de Philosophie Positive," is alike remarkable for its profundity and its shallowness, its truth and its error, its wisdom and its folly, according as it treats of natural and of spiritual things. In all that relates to the former, there is exhibited a breadth of knowledge as to facts, a depth of penetration as to principles, a subtlety in discrimination, a skill in generalization, and withal a facility in expressing truths the most abstruse and profound in language rigidly accurate yet readily intelligible, that has seldom if ever been equalled. All that relates to the latter is but a notable illustration of the language, if not the precise idea of the apostle, "the natural man receiveth not the things of the Spirit of God: for they are foolishness unto him: neither can he know them, because they are spiritually discerned." Remarkable clearness and extent of vision as to natural things is combined with total blindness in regard to all that pertains to man's spiritual nature and relations.

The classification of the Physical Sciences given in "the Positive Philosophy," has been justly styled by Morell, "a master-piece of scientific inquiry." With the skill of an expert topographer, the author presents a synoptical view—alike remarkable for its comprehensiveness and its minuteness—of the whole domain of Physical Science, the proper limits of each department distinctly traced, its peculiar features graphically sketched, and the true relations of the several departments to each other exhibited so clearly as to be apprehended at a glance.

But it is in his analysis of Mathematics—the science which he justly regards as "holding the first place in the hierarchy of the sciences"—the "scientia scientiarum"—that the remarkable powers of his intellect above referred to, are most strikingly illustrated. Strange as it may seem, the task he here undertook to perform was one which previously had scarcely been attempted. The very definition of Mathematics as a distinct science was undetermined, and the accepted classification of its several branches was largely arbitrary and based upon superficial rather than profound logical considerations. The explanation of this fact which Comte himself suggests is, that "the different fundamental conceptions which constitute this great science were not, until the commencement of the last century, sufficiently developed to permit the true spirit of the whole to manifest itself with clearness. Since that epoch the attention of mathematicians has been too exclusively absorbed in the special perfecting of the different branches, and in the application of them to the investigation of the laws of the universe, to allow of due attention to the general system of the science."

Entering upon this almost wholly unexplored field, Comte has succeeded in giving so complete and accurate a survey that little remains for those who come after him but to follow in his footsteps, or, as in the future the domain of the science may be enlarged, to advance along the paths whose direction he has indicated. So original and exhaustive is the work which Comte has here performed, that the language of Mill is not extravagant when he attributes to him the honour of "having created the philosophy of mathematics."

In availing ourselves of the labours of Comte in our attempt to answer the question proposed at the commencement of this paper, we do not feel at liberty to do so without the distinct avowal that however valuable that portion of "the Positive Philosophy" which relates to physical science, we regard it as but a small compensation for the accompanying error which it is the main design of the work to inculcate and to which the author would make all that is really valuable subservient. Weighing the evil against the good, we have no hesitation in deploring, that "this greatest work of the age," as it has been styled by some of its admirers, should ever have been written. The light which it throws upon science could not have been much longer obscured, whilst under the guidance of its teachings in regard to spiritual things many doubtless will be led into regions of everlasting darkness.

What then is Mathematics?

The answer commonly given to this question—and which is objectionable on account of its incompleteness rather than its incorrectness—is, that it is *the science of quantity.* To arrive at a clear idea of the true and complete definition it may be well to start with this defective definition.

Mathematics then is the science of quantity. And what is *quantity?* The etymology of the word indicates its precise meaning. Derived from the Latin, *quantitas*, and that from *quantus—how much?*—the word *quantity* denotes, that which is referred to when we ask with respect to anything, *How much* is there of it? The question does not refer to the form, structure, value, uses, or any other quality or attribute of the thing than its *how-much-ness*, if we may be allowed to coin the synonym. This is the strict and primary meaning of the word *quantity.*

By a very slight metonymy the word is used to denote any *thing*, or any *quality* of any thing, in regard to which the question may be asked, How much is there of it? It is used in this sense when we speak of a quantity or of *quantities*.

The idea expressed by the word *quantity* carries with it the idea of *measurement*. Until a thing is measured it is impossible to give a precise answer to the question, How much is there of it?

And what is meant by *measuring* a quantity? The idea is familiar to all—how may it be defined? It is determining the ratio between the quantity to be measured and some other quantity of the same species, regarded as a unit. The choice of the quantity used as the unit is entirely arbitrary—or rather, conventional. Whatever be its greatness or smallness, if it be of the same species with the quantity to be measured there is between the two a definite ratio, and the determination of this ratio measures the quantity in question. What particular quantity shall be taken as the unit, is a matter of convenience rather than of accuracy.

We are now prepared for a somewhat more precise answer to the question, What is Mathematics? It is that science which has for its object the *measurement* of quantities.

But in many instances measurements may be performed simply by the actual application of the unit of measurement to the quantity to be measured—as, for example, when with a graduated rule we determine that a given line is so many feet and inches in length. This would be the measurement of a quantity, and hence would come within the terms of the above definition, and yet it is evident that such an operation would be purely mechanical—not a scientific, and hence not a mathematical, process. The above answer to the question under consideration needs therefore to be still further amended.

To understand precisely wherein it is defective, and what is the amendment necessary to make it complete, consider, that comparatively few of the almost infinite number and variety of the quantities we may wish to measure admit of measurement by the actual application to them of a unit. Take the simple case of determining the length of a given line—the line may too long or too short to admit of actual measurement; it may be in an inconvenient position; it may be wholly inaccessible; or it may be a curved line, in which case the exact application to it of the linear unit would be impracticable. So to determine the area of a circle, or even of a triangle, the superficial unit being a square, it would be impossible, however small the unit, to apply it actually to the area to be measured. And further, in regard to many quantities the application or superposition of a unit of the same species, is not only impossible but inconceivable, as, for example, such quantities as time, force, velocity, &c.—quantities the most common, involved in many of the most interesting phenomena of nature, and whose exact measure it is often of the highest importance to know.

Now as to all quantities, except the very few that may be measured by the actual application of a unit, how is measurement to be effected? We answer, it can only be done *indirectly*, and indirectly only in this particular way, by means of some definite relation between the quantity to be measured and some other quantity or quantities that admit of actual measurement. For example, to determine the height of a vertical object—if a straight line be measured from the base of the object to any convenient point in the same horizontal plane, and at that point the angle of elevation of the top of the vertical object be measured, its height may be readily determined by means of the measurements made and the definite relation between the height of the object and the quantities actually measured.

We are now prepared to give the precise and complete answer to the question, What is Mathematics? It is that science which has for its object the *indirect* measurement of quantities, that is by means of the relations of the quantity to be measured to some other quantity or quantities that admit of actual measurement.

The object of mathematics as here stated, may seem at first sight to be a very simple one, of comparatively little moment, and requiring for its attainment but a moderate exercise of our intellectual faculties. It needs however but little reflection upon the terms of this definition to enable us to appreciate the comprehensiveness of this science, its immense importance, and the large demand it makes upon the highest powers of the human intellect for its successful prosecution.

As to its comprehensiveness, it has to do with whatever may be called *a quantity*. Its domain therefore is the whole sphere of nature. It includes in its scope the investigation of the form, position, and magnitude of all bodies, their weight, their density, their colour even—for what is colour but the velocity of an undulation? It has to do with the number of every aggregate, the proportions of every chemical combination, the value of every article of commerce, even the pitch of every sound. It is involved in the investigation of all actions of forces, and hence all the phenomena of motion, as well as many of the phenomena of light, of heat, of electricity, of magnetism, of galvanism. It deals with all problems involving the idea of time, for time is but measured duration. In short, it has to do with every thing, and every quality or attribute of every thing, in regard to which the question may be asked, How much is there of it? In the language of the son of Sirach, "God hath made all things by number, weight, and measure." How comprehensive then is that science which includes in its scope whatever may be numbered, weighed, and measured?

To appreciate the *importance* of this science, we have but to consider that until a quantity is measured we cannot have a distinct knowledge of it. Any conception of it that we may previously have, is necessarily incomplete, vague, and for any scientific purpose, valueless. We may, for example, have the idea that the earth as compared to bodies on its surface is large, but we have no proper knowledge of its size until we are able to say it is a sphere of so many miles in diameter. We may know that a body if unsupported will fall toward the earth, but we have no proper, or at least, scientific, knowledge of gravitation until we are able to say that the attraction of matter for matter varies directly as the mass, and inversely as the square of the distance.

We see at once therefore why it is that mathematics lies at the foundation of all true physical science. Without the exact knowledge which it furnishes, the very material for such science will be wanting. Science is facts systematized, that is generalized, and generalization is impossible if our knowledge of the facts themselves be indefinite. It is not an accident, therefore, but a necessary logical consequence that physical science in its progress has always followed and never preceded mathematical science. A complete history of mathematics together with its various applications, would contain a history of the progress of the race in the knowledge of the phenomena of nature.

Certain savage tribes have been found unable to count beyond a hundred. No other fact than this is needed to satisfy us as to their utter mental degradation. It of itself indicates the absence of those precise and accurate conceptions, which both as cause and effect ever accompany intellectual development. For the elevation of such savages, it would be a primary and indispensable requisite that their minds be informed with mathematical ideas, and just in proportion as they should make progress in the knowledge of this science would they be elevated in the scale of intelligence.

By a due consideration of the terms of the above definition of mathematics, it will be further abundantly manifest that it demands the exercise of the highest powers of the human intellect for its successful prosecution. The object of the science, as has been stated, is, the measurement of whatever may be called a quantity, and this, *indirectly*—that is, by means of the relation of the quantity to be measured to other dependent quantities which admit of actual measurement. If then we consider the number and variety of quantities, the exact measure of which it is both interesting and important for us to know; if we consider further how manifold and complex the relations of many quantities to other quantities dependent upon them, and that out of these relations those are to be selected that may be made to answer the end in view; if we consider still further that in many cases the dependent quantities themselves can be measured only indirectly, that is, by other quantities depending upon them; and these again, it may be, only in like manner by others, we may begin to appreciate the magnitude of the difficulty of many of the problems which present themselves to the mathematician for solution.

His first difficulty is to obtain a mathematical expression of the relation of the quantity to be measured to other quantities depending upon it which admit of measurement directly or indirectly. This difficulty having been overcome, another—and in many cases, the greater—difficulty still remains, namely, to reduce the complex expression so that the *precise* mode of dependence of the quantities involved may be exhibited.

The performance of the intellectual task here proposed involves a vigorous exercise of the imagination in the true sense of that oft misused term—the faculty of forming distinct and correct mental images of objects not present to the senses. It requires moreover protracted attention, intense thought, clear conception, and subtle discrimination. Out of all the quantities which are dependent upon, or by the introduction of others, may be brought into relation to, the quantity to be measured, the judgment is exercised in selecting such quantities, and such relations of them, as are suitable to the end in view. The reasoning faculties are exercised in detecting and exhibiting the connection between dependent truths—sometimes the process being synthetic, that is, so *placing together* (*συντιθημι*) known truths as to demonstrate a new truth; sometimes analytic, that is unfolding or *unloosing* (*αναλυω*) the several truths that are involved—wrapped up, as it were—in a general truth or proposition; the former process being analogous to that which is of so much importance in all scientific inquiry—the work of generalization, induction, passing from particulars to generals—the latter, an exercise no less important, that of deduction, or passing from generals to particulars.

It is not strange, therefore, that both in ancient and modern times, men who have been preëminent for superiority of intellect—the master-minds of their age—such men as Pythagoras and Plato, Descartes and Newton—have been attracted to the study of mathematics, and have found therein scope for the exercise of their highest powers. It is not strange that in the progress of the race in intellectual development, the most brilliant achievements—those which most exalt our conception of the capacity and power of the human mind—have been performed on the field of mathematical science. Nor is it strange that a science, the study of which requires the exercise of so many and so important faculties of the mind, and a knowledge of which—to some extent—is indispensable to any intelligent conception of the various phenomena of nature, should occupy the prominent place it does in the course of study pursued by those who seek mental discipline and a liberal education.

Keeping in view the special object of mathematics, as stated in the above definition, we proceed to exhibit the divisions of the science, the distinctive character of each, and their true relation to each other as parts of a logically connected system.

It may be well at this point to define a term which frequently occurs in the philosophy of mathematics, and in a somewhat technical sense—the term *function.* One quantity is said to be a *function* of another when it depends upon the other for its value. The force of a cannon-ball is a function of the quantity of powder and the length of the cannon. The range of the ball is a function of the quantities just mentioned, the elevation of the cannon, the attraction of gravitation, and the resistance of the air. The sine, cosine, &c., of an arc are functions of the arc. The power, root, logarithm, &c. of any number are functions of that number.

Further, functions are said to be *explicit* when the *precise* mode of dependence is expressed; otherwise they are said to be *implicit.* For example, let $x^2+y^2=25$. It is evident that here the value of y depends upon the value of x, but so long as the equation remains in this form, the *precise* mode of dependence is not expressed—the function is implicit. If, however, by the proper algebraic processes the value of y in terms of x be determined, the *precise* mode of dependence would then be expressed—the function would be explicit.

Using then the term function in the sense just mentioned, the complete definition of mathematics previously given may be put in this form—it is the science which has for its object the *indirect* measurement of quantities, that is by means of the relations of the quantity to be measured to some function or functions of it which admit of actual measurement.

By reflecting on what is involved in this definition, it will be manifest that the solution of a complete mathematical problem includes two entirely distinct operations or processes. The first is, determining what function or functions of the quantity to be measured are suitable to the end proposed, and then obtaining a mathematical expression of the relation of the quantity to be measured to the function or functions involved. This result will ordinarily be an implicit function. The second process is, reducing this implicit function to an explicit. The precise relation of the quantity to be measured to quantities that admit of actual measurement will thus be exhibited—that is, the measurement in question will be effected.

These two operations are, as has been said, entirely distinct in character. The first has to do with the nature or species of the quantity in question; also, with the nature or species of the functions involved. The second has nothing to do with the nature or species of the quantity in question or of its functions. It is simply an application of the rules for transforming and reducing an implicit mathematical function; in other words, it has to do, solely, with the relations which are peculiar to *numerical* quantities—the rules referred to being determined entirely by these relations. In reducing the implicit function the process will be the same, whether the quantities involved be lines, surfaces, velocities, forces, or quantities of any species whatsoever. For example, whether the problem involve the proportion that the areas of parallelograms of the same altitude are to each other as their bases, or the proportion that the sides of a plane triangle are to each other as the sines of the angles opposite, or the proportion that the squares of the times of revolution of the planets are as the cubes of their mean distances from the sun, in either case, three of the terms of the proportion being known the fourth would be determined by the same mathematical process.

We have here then a basis for a logical division of the science of mathematics into two great branches, differing from each other as to their immediate objects, methods, and the nature of the quantities involved. The distinctive characteristics of each branch are also clearly indicated.

The division to which the latter operation in the solution of a complete mathematical problem belongs—in logical order the *primary* division, in that it has to do with the relations of numerical quantities only—Comte designates, *Abstract* Mathematics. The other division, which has to do with the relations of quantities of any other species whatsoever, he calls, *Concrete* Mathematics.

It may be proper to remark that this principal division of the science into Abstract and Concrete Mathematics, should not be confounded with the common unscientific division of the science, into Pure and Mixed Mathematics. The two branches in the one case do not correspond either in nature or extent to the two branches in the other, the division in the one case and the other being determined by entirely different considerations. For example, Geometry, according to the ordinary division, is a branch of Pure Mathematics—according to the division abovementioned, it is a branch of Concrete Mathematics. The division of the several branches of the science into Pure and Mixed Mathematics is a kind of mechanical classification, based on superficial considerations. The division into Abstract and Concrete Mathematics is based upon a clear and fundamental distinction between the two branches, as to their elements, methods, and immediate objects.

Before proceeding to specify the *subdivisions* of mathematics, it will be necessary to determine accurately the limitations of the science.

Whilst every conceivable quantity is related to certain other quantities which may be called its functions, it is evident that there are many functions that cannot be used for the object proposed in mathematics, namely, *measurement.* For measurement, as we have already had occasion to remark, being the determination of the ratio of the quantity to be measured to some other quantity of the same species regarded as a unit, the result in every case is a *numerical* quantity. Now this result can never be obtained if the implicit function involve any other relations than those which admit of numerical expression—or expression by means of algebraic symbols of numerical quantities. If other relations than those just mentioned are involved, the processes for transforming and reducing the implicit function—having respect as they have to the relations of numerical quantities only—will be inapplicable. The first limitation therefore to the science of mathematics is, that it is restricted to the use of functions whose mode of dependence admits of numerical expression, or expression by means of algebraic symbols of numerical quantities.

Again, the relations between the quantity to be measured and its functions may admit of algebraic expression, yet so abstruse in character and complicated in form that the reduction to an explicit function cannot, in the present state of mathematical science, be effected. This gives rise to another limitation—if not in theory, at least in practice. Only such quantities are practically within the scope of the science as have functions whose relations admit of a reducible algebraic expression.

To indicate therefore the limits of the science positively, it will be necessary to determine what are the different relations that may exist between *numerical* quantities—in other words, what are the different ways in which one number may depend upon other numbers for its value. The different forms of *implicit* numerical functions are infinite in number and variety. An implicit function is but a combination of several simple or elementary functions, and there is evidently no limit to the number and variety of such combinations. The number of simple or elementary numerical functions, however, is quite limited. In the present state of mathematical science there are but ten simple, abstract functions—strictly speaking, but five, since the ten referred to are in fact five *pairs* of functions, one of each pair being the inverse of the other. Moreover, the last pair are not purely abstract. In certain respects they are of the nature of abstract functions or may be treated as such; in other respects they are concrete.

The five pairs of simple or elementary abstract functions are: 1st. The Sum and Difference; 2d. The Product and Quotient; 3d. The Power and Root; 4th. Logarithmic and Exponential functions; 5th. Direct and Inverse Trigonometrical—or, as they are frequently called, Circular—functions. All possible relations of numerical quantities—or rather all relations that in the present state of mathematical science admit of algebraic expression—are but combinations more or less complicated of these ten simple functions.

The science of Mathematics is, therefore, for the present at least, limited to the investigation of quantities having functions which are dependent in some one or other of the ten ways just mentioned. Even with this limitation it need scarcely be said, that no other science is to be compared with mathematics, with respect to the extent of the field which it embraces.

We are now prepared to consider the subdivisions into which the two great branches of mathematics is divided.

The *subdivisions* which Comte proposes, are—like his principal divisions—determined by an analysis of the solution of a complete mathematical problem. Fundamental distinctions having respect to the nature and especially to the particular object of the different processes involved, are made the basis of classification. He accordingly divides Abstract Mathematics into two branches, which may be designated as Arithmetic and Algebra, if we are careful not to confound the particular sense in which these terms are here employed, with the indefinite, unscientific sense in which they are ordinarily used.

To present clearly the distinction between these two branches, and the precise nature and object of each, suppose it be required to determine the number which being multiplied by 3 will produce 12. The problem may be solved by writing 12 and dividing it by 3; or we may write $x =$ the number required—then by the hypothesis $3\,x = 12$, hence $x = 4$. Now according to the ordinary acceptation of the terms the former operation would be called *arithmetical*—the latter, *algebraic.* And yet it is evident that the two operations are essentially the same. They have precisely the same object, and this object is reached by precisely the same mental process in either case. The two operations differ only in form, that is in appearance. The distinction between them is therefore entirely superficial, and does not furnish a sufficient basis for a truly scientific classification.

Suppose, however, that the *general* problem had been given, to find an expression for the number which multiplied by a shall produce b?—a and b denoting any numbers whatsoever. Here if we denote the required number by x, then by the hypothesis $ax = b$ and therefore $x = b$ divided by a. Now here whilst the *form* of the process is identical with the preceding, the *object* and the *result* are entirely different. The result $x = b$ divided by a, is not a *value* of x, it merely exhibits *the mode of dependence* of x on a and b—in other words, it exhibits *what* function x is of a and b, whatever values be attributed to these latter symbols. If we wish to know the particular value of x, when $b = 12$ and $a = 3$, it may readily be determined by substituting the given values in the general expression and performing the division indicated. This operation is entirely different from the preceding both as to its nature and its object. In these respects it corresponds exactly with the problem first proposed.

We have here then the distinction between Arithmetic and Algebra, in the strictly scientific sense of those terms, clearly indicated. That part of Abstract Mathematics which has for its object to determine *what precise function* one quantity is of another or of others with which it has the relations expressed in the conditions of the problem, is Algebra. That part of Abstract Mathematics which has for its object to determine *the precise numerical value* of the quantity in question, is Arithmetic. The former has respect only to the *relations* of the quantities involved, the latter has respect to their *values*. It may be remarked that as every purely algebraic problem includes all possible similar arithmetical problems, so every purely arithmetical problem may be regarded as but a particular case of a general algebraic problem.

As the terms Arithmetic and Algebra are ordinarily used in a less strict sense than that above-mentioned, and moreover are not etymologically significant, the two branches of Abstract Mathematics may be more precisely designated as the Calculus of Values, and the Calculus of Functions.

Arithmetic, or the Calculus of Values, from its very nature, does not admit of logical subdivision; not so, however, with Algebra or the Calculus of Functions.

We have seen that the number of simple abstract functions—that is, those which may exist between numerical quantities—is very limited. On the other hand, the number and variety of concrete functions is so great that they may be said to be unlimited. Now, as the ultimate object of mathematics is the *measurement* of quantities of any species whatsoever, and as measurement is the determination of the *ratio* of two quantities of the same species, and, moreover, as every such ratio is a *numerical* quantity, it is evident that only such concrete functions can be employed as have relations which admit of algebraic expression—an expression, moreover, that is reducible to an *explicit* function, and to an explicit function of such a form that its numerical value for given values of the symbols involved may be obtained by the processes of arithmetic. It follows, therefore, as has been before remarked, that the selection of appropriate concrete functions and the obtaining a suitable expression for their relations, often presents to the mathematician a most formidable difficulty.

It might seem at first sight that this difficulty might be lessened by simply increasing the number of elementary abstract functions. A little reflection will show that this is almost wholly impracticable. The possible relations of numerical quantities is evidently quite limited, and whilst we may not say that in the future progress of mathematical science no new abstract functions will be recognized, it is difficult for us at present to conceive of any others—at least any others available for the end in view—than the ten simple functions above mentioned.

The difficulty in question has however been very ingeniously encountered in another way, namely, not by attempting to increase the number of simple functions, but by making use of certain *functions of these functions.* Theoretically there are several different functions of the simple functions that might be employed, but it is found that practically the most suitable by far for the end in view, are the infinitesimals or differentials of the simple functions. By *an infinitesimal* (according to the theory of Leibnitz, which, though not so rigidly accurate, is more readily intelligible than the theories of Newton and Lagrange,) is meant a portion of a quantity less than any assignable fraction of it—in that sense, *infinitely* small—relatively, though not absolutely, equal to zero. A *differential*—which in the theory of Leibnitz corresponds to what Newton calls a *fluxion*, and Lagrange a *derivative*—is the infinitesimal of a *variable*, or the difference between two successive values of a variable.

It is found that in dealing with many of the most interesting yet otherwise insoluble problems of Concrete Mathematics, these infinitesimals are admirably adapted to overcome the difficulty referred to above. Relations of Concrete Functions, which it would be impossible to express immediately in terms of Abstract Functions, may frequently be readily expressed in terms of the differentials of Abstract Functions. From the equation thus obtained—that is, an equation which expresses the relation between the *differentials* of Abstract Functions—an equation expressing the relation between *the abstract functions themselves* may, by established rules for such transformation, be readily obtained—thus bringing the quantity in question within the grasp, as it were, of that branch of the science by which alone its measurement can be effected.

As the true scientific conception of the Infinitesimal Calculus is that which has just been presented, it may be well to illustrate the idea by a simple example. Suppose the problem to determine the area of a plane curve. Now, whilst the area of some few curves may be obtained by special processes, different in the case of different curves and always cumbersome in application, no general method—applicable to any and every curve—involving the use of the functions of the curve *directly*, can be given. By using, however, the differentials of the quantities involved, the *differential* of the area may be expressed by a very simple formula, namely, the differential of the area is equal to one rectilinear ordinate of the curve into the differential of the other. By means of this formula an expression for the *differential* of the area of any given curve may be readily obtained. Then by strict mathematical processes this equation in terms of the differentials, may be transformed, and the value of the area itself, in terms of one of its functions exhibited.

It should be remarked that the solution of a number of the most interesting problems of mathematics is immediately effected, whenever an equation expressing the relation of the differentials of the variables involved is obtained. For example, to determine the angle which a given curve at a given point makes with the abscissa or abscissa produced—the ratio of the differential of the ordinate of the point to the differential of the abscissa expresses at once the tangent of the angle required.

From what has been said, the logical division of Algebra, or the Calculus of Functions is evident. It is divided into two branches—one deals with functions themselves and hence involves finite quantities only, the other deals with the differentials of functions and hence involves infinitesimals. The former is ordinary Elementary Algebra, the latter Transcendental

Algebra. To distinguish these branches by designations that shall be significant, the former may be called the Calculus of Direct Functions, the latter, the Calculus of Indirect Functions.

The Calculus of Indirect Functions includes two entirely different mathematical processes, the one the inverse of the other and yet its logical complement in a conception of this branch of mathematical science. These processes are known as differentiation and integration. The relation of two quantities which are functions of each other being given, *differentiation* is determining the relation of the differentials of these quantities. Again, the relations of the differentials of two quantities—functions of each other—being given, *integration* is determining the relation of the quantities themselves.

The Differential *Calculus* in the strict sense of the expression—namely, that branch of Abstract Mathematics which has for its object the differentiation of any given function—is quite limited in its scope, and may be said to have reached its perfection. Convenient rules for the differentiation of all the recognized simple abstract functions have been determined; and any complex function, being but a combination of simple functions, may always be differentiated by applying successively the differentiation of the several simple functions involved. The *application* of the Differential Calculus, however, to the solution of problems in Concrete Mathematics, is unlimited in extent.

The scope of Integral Calculus proper—that is the integration of any given relation of differentials—is much wider, and its development is comparatively quite imperfect. The limits of this branch of the Calculus of Indirect Functions are—and probably must always remain—indefinite. A relation of the differentials of functions may be given such that the relation of the functions themselves will not admit of algebraic expression. Or the immediate result of the differentiation of a complex function may be so transformed by legitimate algebraical processes that the derivation may be entirely obscured.

To complete our synopsis of Abstract Mathematics two Calculi remain to be noticed. First, the Calculus of Variations, invented by Lagrange and largely used by him in his "Analytique Mechanique." This branch of mathematics Comte fitly characterizes as "hyper-transcendental." It is so abstruse in its nature and complicated in its processes that it has received but little attention from mathematicians, and remains in about the same state in which Lagrange left it. The object of the Calculus of Variations, as stated by Comte, is, to determine "what form a certain unknown function of one or more variables ought to have, in order that the value of a given integral within assigned limits shall, for that function, be a maximum or minimum in comparison with the values of the integral for functions of any other form whatsoever." What is meant by this, the reader who has some knowledge of the Differential and Integral Calculus may understand by the following illustration. In an ordinary problem in Maxima and Minima, a function is given, and it is proposed to determine what is that value of the variable which will render the function a maximum or minimum in comparison with either of the values immediately adjacent, that is in comparison with either of the values the function would have if the value of the variable were either increased or diminished. The Differential Calculus furnishes a ready method for the solution of all such problems. Now suppose instead of a function being given, a general expression for the value of the integral of a certain differential expression is given —for example, the area of a plane curve = the integral of ydx—and it is proposed to determine the equation of the curve whose area is a maximum in comparison with the area of any other curve between the same limiting ordinates,—in other words, what must be the relation of the function y to the variable x that the integral of ydx shall be a maximum. This problem is of an entirely different nature from an ordinary problem in Maxima and Minima. Here there is no function given—the very problem is to determine the function, in other words, the equation of the curve whose area is a maximum. Moreover, the maximum referred to in an ordinary Maxima and Minima problem is that value of the function which is a maximum as compared with either of the values the function would have if the value of the variable were either increased or diminished. Here the maximum referred to is a maximum, not as compared with the values which the quantity in question would have if changes were attributed to the variable, but a maximum as compared with the value which the quantity in question would have if any change should be made in the form of the function. It is with the class of problems of which the one just mentioned is a simple example, that the Calculus of Variations deals.

The remaining branch of Abstract Mathematics above referred to, is the Calculus of Finite Differences, invented by Taylor and exhibited in his "Methodus Incrementorum." This Calculus is but an extension of the fundamental idea—or rather a new application of the *method*—of the Differential and Integral Calculus. This latter has for its primary object to determine the change in any function corresponding to an *infinitely small* change in the variable upon which it depends. The immediate object of the Calculus of Finite Differences is to determine the change in any function corresponding to a particular *finite* change in the variable; or more generally, to determine the successive changes in any function corresponding to successive finite changes attributed to the variable according to a given law—as for example, when the values of the variable increase (as they are ordinarily, in this Calculus, assumed to do) in Arithmetical Progression. Like the Infinitesimal Calculus, the Calculus of Finite Differences has two branches—the Direct and the Inverse—the latter being sometimes called the Integral Calculus of Finite Differences. In its form and notation the Calculus of Finite Differences is analogous to the Calculus of Indirect Functions, yet as it deals entirely with *finite* quantities it is logically a branch of the Calculus of Direct Functions.

Having completed our survey of Abstract Mathematics, it may not be amiss to give a summary of the points that have been presented.

Mathematics is that science which has for its object the indirect measurement of quantities, that is by means of the relations of the quantity to be measured to other dependent quantities—called its functions—which admit of actual measurement.

The science is divided into two branches—Abstract Mathematics which treats of the functions of numerical quantities, and Concrete Mathematics which treats of functions of any other species. Abstract Mathematics is divided into two branches—Arithmetic or the Calculus of Values, and Algebra or the Calculus of Functions.

The Calculus of Functions is divided into two branches—the Calculus of Direct Functions or ordinary Elementary Algebra, which deals with finite quantities only; and the Calculus of Indirect Functions, which investigates the relations of infinitesimal quantities.

The Calculus of Indirect Functions is divided into two branches—the Differential and Integral Calculus, and the Calculus of Variations.

The Calculus of Finite Differences investigates the relations of corresponding finite increments of functions, by the methods of the Infinitesimal Calculus.

It remains for us to notice—which we can do only in a very summary manner—the Philosophy of Concrete Mathematics. As this branch of the science has to do with functions of any species whatsoever, its scope, in theory at least, is coextensive with the material universe—or rather, with so much of the material universe as lies within the sphere of our knowledge. As the phenomena of nature are infinite—not only in number but in variety—it might seem at first sight that any subdivision of Concrete Mathematics into distinct branches would not—however great their number—be exhaustive. Upon a more profound view however of the functions with which this branch of mathematical science has to deal, it will appear that there are but two really distinct divisions of Concrete Mathematics.

The phenomena of the material universe, however manifold and varied in form and appearance, are all ultimately resolvable into two constituent elements, so to speak, namely, matter and force. Every particular phenomenon is but a particular modification or combination of these elements.

Whilst we know not what *matter* is in its essence, it may nevertheless be defined as that which occupies space—in other words, a distinctive essential property of it is *extension.* Whilst we know not what *force* is in its essence, it may be defined as that which produces, modifies, prevents, or tends to produce modify or prevent, *motion.* It is evident therefore that Geometry—the science which has for its object the measurement of extension—and Rational Mechanics—the science which has for its object the measurement of the action of forces—include, theoretically, within their scope all the problems of Concrete Mathematics. If the material universe were immovable the only phenomena of nature would be the magnitude, form, and position of bodies—that is, the problems of Natural Philosophy would be exclusively geometrical. As the universe is constituted, however, matter is continually subject to the action of forces—the phenomena therefore actually presented involve mechanical as well as geometrical problems.

Of these two divisions of Concrete Mathematics, the primary—in logical order as well as with respect to the simplicity of its elements—is Geometry. To understand the true spirit of this branch of Mathematics—which has for its object the measurement of extension—and to appreciate the bearing and ultimate destination of all geometrical inquiries, it should be remarked, that extension may be in *one* direction, in *two* directions, or in *three* directions. Geometry accordingly has to deal with three entirely different kinds of quantity, namely, lines, surfaces and—what are popularly called, solids—in the more exact language of science, volumes. The limit of any material body is a surface; the limit of a surface or the intersection of two surfaces is a line; the limit of a line or the intersection of two lines is a point. A point having position only and not magnitude, does not admit of measurement. A more explicit definition therefore of Geometry than that given above would be, it is that branch of Mathematics which has for its object the measurement of lines, surfaces, and volumes.

These quantities are not only different in kind, but the kind of *measurement* of which they are severally susceptible is entirely different. Under favourable circumstances a line may be measured *directly*, that is, by the application to it of a unit of measure. The direct application of a unit of measure to a *surface* is ordinarily impracticable; to a *volume* it is ordinarily—from the very nature of the case—impossible. The measurement of surfaces and volumes therefore is to be effected only *indirectly*, that is, by means of their relations to some quantities that admit of actual measurement. Now the magnitude of any surface or volume is always dependent upon the magnitude of certain *lines* pertaining to it, for example, the area of an ellipse upon the lengths of its principal axes, the capacity of a cylinder on the diameter of its base and its height. The general object therefore of Geometry as it respects surfaces and volumes, is to determine a definite relation between the quantity in question and some linear function of it.

But further, as to the measurement of *lines*—whilst a *straight* line under favourable circumstances may be measured by the application to it of a unit, if a line be *curved* the exact application to it of a rectilinear unit is impracticable; its measurement accordingly can be effected only indirectly, that is, by means of its relations to some straight line that admits of measurement either directly or indirectly. The general object therefore of Geometry as it respects *curved* lines, is to determine some definite relation between the line in question and some *straight* line.

Once more, but few of the *straight* lines whose measure we may wish to know admit of measurement by the application of a unit. The general object therefore of Geometry with respect to *straight* lines is to determine some definite relation between the line in question and some other straight line that admits of direct measurement. To this precise destination then all geometrical inquiries tend. This ultimate object of the science is the immediate object of that branch of it called Trigonometry. From the relations of the sides and angles of a plane triangle, simple rules are established by means of which any three parts—one being a side—of a plane triangle being known the other three may be determined. Hence to measure indirectly any given straight line, it is only necessary to regard it as a side of a triangle of which three parts admit of actual measurement.

The above definition of Geometry may at first sight seem to be defective, inasmuch as a large part of the actual science is the investigation of the *properties* of lines—not their measurement. By taking, however, a comprehensive view of the whole subject, the important bearing of such investigations on what we have stated to be the true object of Geometry, may readily be traced.

All properties of a line, a surface, or a volume, are not equally suitable for the purpose of its measurement. Some properties—and it may be those most readily recognized—are wholly unsuitable. Had Archimedes known no other property of the parabola than that it was a section of a cone parallel to the opposite slant side, he would never have been able to effect its quadrature. It is evident, therefore, that just in proportion as the number of known properties of a line is increased, its rectification, quadrature, and the curvature of the volume generated by its revolution, will be facilitated.

Again, the ultimate object of geometrical science is the measurement of material bodies as they exist in nature—that is, the measurement of *concrete* lines, surfaces, and volumes. To effect, however, the measurement of any given concrete line, surface, or volume, its correspondence or similarity to some one or other of the theoretical lines, surfaces, volumes—the abstract types, so to speak—which Geometry has investigated and determined a method for the measurement, must first be recognized. For example, the size of the earth could not possibly have been determined before it was known that the earth was—approximately at least—a sphere. Now the correspondence of any given concrete geometrical quantity to some particular abstract type can be detected only by recognizing the existence of some characteristic property common to both. Sometimes this correspondence is detected by means of one property of the type, sometimes by means of another. It is evident, therefore, that just in proportion as the number of known properties of the several lines and surfaces investigated by Geometry is increased, will the recognition of the similarity of any given concrete quantity to its corresponding abstract type be facilitated; or, as it is expressed by Comte, "the study of the *properties* of lines and surfaces is indispensable to organizing in a rational manner the abstract and the concrete in geometry."

An interesting illustration of the above is furnished by Kepler's memorable discovery that the orbits of the planets are elliptical. Had he known no other properties of an ellipse than that it is an oblique section of a cone, or that the sum of the distances of any point on the curve from two fixed points is constant, he would never have been known as "the Legislator of the Heavens." But observing that the relation of the distance of Mars from the sun to the direction of the planet was the same as the relation of the length of a radius vector of an ellipse to its direction from the focus, the character of the orbit was indicated, and once indicated the suggestion was soon incontrovertibly confirmed.

As to the subdivisions of Concrete Mathematics—Geometry is divided into two branches, Synthetic (or Elementary) Geometry and Analytical Geometry. The characteristic distinction between these two branches, as to their *methods*, is indicated by their respective names. The method of the former is the demonstration of a new geometrical truth by the synthesis or combination of truths previously known. The latter is not simply the application of Algebra to Geometry—algebraic characters may readily be used in strictly synthetic demonstration. Analytical Geometry in the strict and proper sense is that particular use of algebraic symbols which consists in representing a line (or surface) by an algebraic equation expressing the relation between the variable functions of the line (or surface) and then determining the properties of the line (or surface) by an analysis of its equation. Synthetic Geometry always deals *directly* with the line or surface investigated; Analytical Geometry investigates the quantity in question *indirectly*, that is by means of its functions.

There is, however, a still more fundamental distinction between these two branches of mathematical science. Geometry, in theory at least, includes within its scope all imaginable figures, and all the properties of each. In view of the distinction between these two classes of subjects into which the material, so to speak, of the science is divided, it is evident that in the study of Geometry two different plans of procedure may be pursued. One plan would be to study each geometrical *figure* separately and independently—determining all its properties, without any consideration of other figures, even though they might have many analogous properties. The other plan would be to study, separately and independently, each geometrical *property*, determining all the figures which have this property in common, and investigating its peculiarities in each, without any consideration of other properties of the figure in question.

The result, according to the former plan of procedure, would be to exhibit the whole body of geometrical truth as made up of a series of *groups* of facts, having no logical connection with each other. According to the latter plan, the phenomena, so to speak, of the science, would be generalized, and the whole body of truth exhibited in a systematic form. It is scarcely necessary to say which of these two plans of procedure is the more truly scientific. The former was the plan adopted by the ancients, and is a distinctive characteristic of Synthetic Geometry. The latter is that which has been pursued by the moderns since the time of Descartes, and is a distinctive characteristic—and the most fundamental one—of Analytical Geometry. In text-books on Geometry—where the difference between these two branches with respect to *method* is the distinction which should be made most prominent—the ordinary designations are to be preferred. In the Philosophy of Mathematics—where the more fundamental distinction should be made the more prominent—the appropriate designations of the two branches of Geometry are, Special and General Geometry.

The limits of our paper forbid any more extended notice of the philosophical character of the subdivisions of Geometry, as well as any attempt to exhibit the Philosophy of the other principal division of Concrete Mathematics—Rational Mechanics. For a full discussion of Analytical Geometry—that branch of Mathematics, the invention of which marks a new era in the history of physical science, we might say, in the history of the intellectual development of the race—we refer our readers to a very able and interesting Article from the pen of the late Professor Dod, published in the October number of this journal for the year 1841. We would also commend to the notice of such of our readers as are especially interested in mathematical studies, an admirable translation of so much of Comte's "Cours de Philosophie Positive" as relates to the Philosophy of Mathematics, by Professor Gillespie, of Union College, published by the Harpers.